PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Mike Bennett, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before July 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11558. *Proposed by Andrew McFarland, Płock, Poland.* Given four concentric circles, find a necessary and sufficient condition that there be a rectangle with one corner on each circle.

11559. *Proposed by Michel Bataille, Rouen, France.* For positive p and $x \in (0, 1)$, define the sequence $\langle x_n \rangle$ by $x_0 = 1$, $x_1 = x$, and, for $n \geq 1$,

$$x_{n+1} = \frac{p x_{n-1} x_n + (1-p) x_n^2}{(1+p) x_{n-1} - p x_n}.$$

Find positive real numbers α, β such that $\lim_{n\to\infty} n^\alpha x_n = \beta$.

11560. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.*
(**a**) The diagonals of a convex pentagon $P_0P_1P_2P_3P_4$ divide it into 11 regions, of which 10 are triangular. Of these 10, five have two vertices on the diagonal P_0P_2. Prove that if each of these has rational area, then the other five triangles, and the original pentagon, all have rational areas.
(**b**) Let $P_0, P_1, \ldots, P_{n-1}$, $n \geq 5$ be points in the plane. Suppose no three are collinear, and, interpreting indices on P_k as periodic modulo n, suppose that for all k, $P_{k-1}P_{k+1}$ is not parallel to P_kP_{k+2}. Let Q_k be the intersection of $P_{k-1}P_{k+1}$ with P_kP_{k+2}. Let α_k be the area of triangle $P_kQ_kP_{k+1}$, and let β_k be the area of triangle $P_{k+1}Q_kQ_{k+1}$. For $0 \leq j \leq 2n-1$, let

$$\gamma_j = \begin{cases} \alpha_{j/2}, & \text{if } j \text{ is even;} \\ \beta_{(j-1)/2}, & \text{if } j \text{ is odd.} \end{cases}$$

Interpreting indices on γ_j as periodic modulo $2n$, find the least m such that if m consecutive γ_j are rational, then all are rational.

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11561. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania.* Let $f_1, \ldots, f_n$ be continuous real valued functions on $[0, 1]$, none identically zero, such that $\int_0^1 f_i(x) f_j(x)\, dx = 0$ if $i \neq j$. Prove that

$$\prod_{k=1}^{n} \int_0^1 f_k^2(x)\, dx \geq n^n \left(\prod_{k=1}^{n} \int_0^1 f_k(x)\, dx \right)^2,$$

$$\sum_{k=1}^{n} \int_0^1 f_k^2(x)\, dx \geq \left(\sum_{k=1}^{n} \int_0^1 f_k(x)\, dx \right)^2, \quad \text{and}$$

$$\sum_{k=1}^{n} \frac{\int_0^1 f_k^2(x)\, dx}{\left(\int_0^1 f_k(x)\, dx \right)^2} \geq n^2.$$

11562. *Proposed by Pál Péter Dályay, Szeged, Hungary.* For positive a, b, c, and z, let $\Psi_{a,b,c}(z) = \Gamma((za + b + c)/(z + 2))$, where Γ denotes the gamma function. Show that $\Psi_{a,b,c}(z)\Psi_{b,c,a}(z)\Psi_{c,a,b}(z)$ is increasing in z for $z \geq 1$.

11563. *Proposed by Vlad Matei (student), University of Bucharest, Bucharest, Romania.* For each integer $k \geq 2$, find all nonconstant f in $\mathbb{Z}[x]$ such that for every prime p, $f(p)$ has no nontrivial kth-power divisor.

SOLUTIONS

Explaining a Polynomial

11403 [2008, 949]. *Proposed by Yaming Yu, University of California–Irvine, Irvine, CA.* Let n be an integer greater than 1, and let f_n be the polynomial given by

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x + j).$$

Find the degree of f_n.

Solution by Nicolás Caro, IMPA, Rio de Janeiro, Brazil, and independently by Cosmin Pohoata, Tudor Vianu National College, Bucharest, Romania. The degree of f_n is $\lfloor n/2 \rfloor$. This follows immediately from the stronger statement that the coefficient of x^r in $f_n(x)$ is the number of derangements of $[n]$ with r cycles, since each cycle must have at least two elements. Here $[n] = \{1, \ldots, n\}$, and a derangement is a permutation with no fixed points.

Let $c(n, k)$ be the number of permutations of $[n]$ with k cycles (the unsigned Stirling number of the first kind). The well-known generating function for these numbers is given by $\sum_{k=1}^{n} c(n, k)x^k = \prod_{j=0}^{n-1}(x + j)$ (provable in many ways, including induction on n). Thus

$$f_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \sum_{k=1}^{i} c(i, k)x^k = \sum_{\ell=0}^{n} \binom{n}{\ell} (-x)^{\ell} \sum_{k=0}^{n-\ell} c(n - \ell, k)x^k$$

$$= \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^{\ell} \sum_{r=\ell}^{n} c(n-\ell, r-\ell)x^r = \sum_{r=0}^{n} \sum_{\ell=0}^{r} \binom{n}{\ell} (-1)^{\ell} c(n-\ell, r-\ell)x^r.$$

The coefficient of x^r in this expression is precisely the inclusion-exclusion formula to count permutations with r orbits that have no fixed points. The universe is the set of permutations with r orbits, and the ith of the n sets to be avoided is the set of permutations in which element i is a fixed point.

Editorial comment. Let $d(n, r)$ be the number of derangements of $[n]$ with r cycles. The fact that the coefficient of x^r in $f_n(x)$ is $d(n, r)$ can also be proved by induction on n using Pascal's formula and the recurrence $d(n, r) = (n-1)[d(n-2, r-1) + d(n-1, r)]$.

O. P. Lossers (and others) gave a short proof of the degree statement by observing that $f_n(x)$ is the nth derivative (with respect to t) of the product $e^{-tx}(1-t)^{-x}$, evaluated at $t = 0$.

This polynomial appears explicitly in *An Introduction to Combinatorial Analysis*, chapter 4 section 4, pp. 72–74 by Riordan (Wiley, 1958). It is also mentioned in *Advanced Combinatorics*, chapter VI, section 6.7, p. 256, by Comtet (Reidel, 1974) with some references to previous non-combinatorial appearances in articles by Tricomi and Carlitz.

Also solved by T. Amdeberhan & S. B. Ekhad, R. Bagby, D. Beckwith, R. Chapman (U.K.), P. Corn, P. P. Dályay (Hungary), O. Geupel (Germany), D. Grinberg, J. Grivaux (France), S. J. Herschkorn, E. Hysnelaj (Australia) & E. Bojaxhiu (Albania), D. E. Knuth, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), Á. Plaza & S. Falcón (Spain), M. A. Prasad (India), R. Pratt, O. G. Ruehr, B. Schmuland (Canada), A. Stadler (Switzerland), J. H. Steelman, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), BSI Problems Group (Germany), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

Mean Inequalities

11434 [2009, 463]. *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.* Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $x_1, \ldots, x_n$ be distinct real numbers, and let $p_1, \ldots, p_n$ be positive numbers summing to 1. Let

$$S = \frac{\sum_{k=1}^{n} p_k x_k^3 - \left(\sum_{k=1}^{n} p_k x_k\right)^3}{3\left(\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{k=1}^{n} p_k x_k\right)^2\right)}.$$

Show that $\min\{x_1, \ldots, x_n\} \leq S \leq \max\{x_1, \ldots, x_n\}$.

Solution by Jim Simons, Cheltenham, U.K. Consider a probability distribution on the real line that takes value x_j with probability p_j for $1 \leq j \leq n$. Write μ_i' for the ith moment about 0 and μ_i for the ith moment about the mean μ_1'. Now

$$S = \frac{\mu_3' - {\mu_1'}^3}{3(\mu_2' - {\mu_1'}^2)} = \frac{\mu_3 + 3\mu_1'\mu_2}{3\mu_2} = \mu_1' + \frac{\mu_3}{3\mu_2}.$$

From this we obtain inequalities stronger than those proposed:

$$\frac{1}{3}\min\{x_1, \ldots, x_n\} + \frac{2}{3}\mu_1' \leq S \leq \frac{1}{3}\max\{x_1, \ldots, x_n\} + \frac{2}{3}\mu_1'.$$

Also solved by R. Bagby, R. Chapman (U.K.), M. P. Cohen, W. J. Cowieson, P. P. Dályay (Hungary), H. Dehghan (Iran), P. J. Fitzsimmons, D. Grinberg, E. A. Herman, T. Konstantopoulos (U.K.), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Posch, K. Schilling, B. Schmuland (Canada), R. Stong, M. Tetiva (Romania), B. Tomper, BSI Problems Group (Germany), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

A Circumradius Equation

11443 [2009, 548]. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.* Consider a triangle ABC with circumcenter O and circumradius R. Denote the distances from O to the sides AB, BC, CA, respectively, by x, y, z. Show that if ABC is acute then $R^3 - (x^2 + y^2 + z^2)R = 2xyz$, and $(x^2 + y^2 + z^2)R - R^3 = 2xyz$ otherwise.

Solution by Philip Benjamin, Berkeley College,Woodland Park, NJ. We first prove the identity

$$1 - \left(\cos^2 A + \cos^2 B + \cos^2 C\right) = 2\cos A \cos B \cos C. \qquad (*)$$

Indeed, $C = \pi - (A + B)$, so $\cos C = -\cos(A + B) = \sin A \sin B - \cos A \cos B$. Isolating $\sin A \sin B$ and squaring yields $\cos^2 A \cos^2 B + 2\cos A \cos B \cos C + \cos^2 C = \sin^2 A \sin^2 B = 1 - \cos^2 A - \cos^2 B + \cos^2 A \cos^2 B$. This simplifies to $(*)$.

Let the side lengths a, b, and c be opposite angles A, B, and C, respectively, so $a/\sin A = b/\sin B = c/\sin C = 2R$. The perpendicular from O to side AB bisects AB, so we have a right triangle with side lengths x, $c/2$, and R. Since $c = 2R\sin C$, we conclude that $x = R|\cos C|$. Similarly $y = R|\cos A|$ and $z = R|\cos B|$. If $\triangle ABC$ is acute, then the three cosines are positive, so multiplying $(*)$ by R^3 produces the desired result. Otherwise, say angle C is right or obtuse. Now $x = -R\cos C$ and the other two cosines are positive. Again, multiplying $(*)$ by R^3 produces the desired result.

Editorial comment. A similar problem was proposed in *Crux Mathematicorum with Mathematical Mayhem,* December, 2008, Problem 3395.

Also solved by A. Alt, H. Bailey, M. Bataille (France), D. Beckwith, R. Chapman (U.K.), L. Csete (Hungary), C. Curtis, P. P. Dályay (Hungary), P. De (India), D. Fleischman, V. V. García (Spain), M. Garner & J. Zacharias, O. Geupel (Germany), M. Goldenberg & M. Kaplan, M. R. Gopal, D. Gove, J.-P. Grivaux (France), L. Herot, J. G. Heuver (Canada), E. Hysnelaj (Australia) & E. Bojaxhiu (Germany), Y. K. Jeon (Korea), G. A. Kandall, Y. H. Kim (Korea), L. R. King, B. Klotzsche, T. Konstantopoulos (U.K.), O. Kouba (Syria), K.-W. Lau (China), J. C. Linders (Netherlands), J. H. Lindsey II, O. P. Lossers (Netherlands), J. McHugh, J. Minkus, J. H. Nieto (Venezuela), P. Nüesch (Switzerland), J. Oelschlager, M. Omarjee (France), J. Posch, C. R. & S. Selvaraj, R. A. Simon (Chile), J. Simons (U.K.), R. Stong, M. Tetiva (Romania), B. Tomper, Z. Vörös (Hungary), M. Vowe (Switzerland), H. Widmer (Switzerland), S. Xiao (Canada), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

Extrema

11449 [2009, 647]. *Proposed by Michel Bataille, Rouen, France.* (corrected) Find the maximum and minimum values of

$$\frac{(a^3 + b^3 + c^3)^2}{(b^2 + c^2)(c^2 + a^2)(a^2 + b^2)}$$

given that $a + b \geq c > 0$, $b + c \geq a > 0$, and $c + a \geq b > 0$.

Solution by Jim Simons, Cheltenham, U.K. Call this big expression X. Since X is homogeneous, we may assume $a^2 + b^2 + c^2 = 1$. The feasible region then consists of a triangular patch on the positive octant of the unit sphere, excluding the vertices (where one of a, b, c is zero), but including the interiors of the sides (where two of a, b, c are equal). Using spherical polar coordinates, we may set $(a, b, c) =$

$(\cos\alpha, \sin\alpha\cos\theta, \sin\alpha\sin\theta)$, where, since a, b, c are positive, θ is uniquely determined and $0 < \theta < \pi/2$. Now

$$X = \frac{\left(\cos^3\alpha + \sin^3\alpha(\cos^3\theta + \sin^3\theta)\right)^2}{\sin^2\alpha\left(\cos^2\alpha + \sin^2\alpha\sin^2\theta\right)\left(\cos^2\alpha + \sin^2\alpha\cos^2\theta\right)}$$
$$= \frac{\left(\cos^3\alpha + \sin^3\alpha(\cos^3\theta + \sin^3\theta)\right)^2}{\sin^2\alpha\left(\cos^4\alpha + \cos^2\alpha\sin^2\alpha + \sin^4\alpha\sin^2\theta\cos^2\theta\right)}.$$

If $f(\theta) = \cos^3\theta + \sin^3\theta$, then $f'(\theta) = 3\cos\theta\sin\theta(\sin\theta - \cos\theta)$. In the feasible region for θ, this is positive for $\theta > \pi/4$ and negative for $\theta < \pi/4$. Thus f, and with it, the numerator of X for fixed α, is less at $\theta = \pi/4$ than at any other feasible θ. Similarly, if $g(\theta) = \sin^2\theta\cos^2\theta$, g, and with it, the denominator of X for fixed α, is increasing in θ for $\theta < \pi/4$ and decreasing in θ for $\theta > \pi/4$. Thus X is, for fixed α, smallest at $\theta = \pi/4$, and greatest at an edge of the feasible region. By symmetry, the minimum value of X is $9/8$, attained when $a = b = c$.

From the foregoing, the maximum value of X on the closure of the feasible region occurs at a point where, with respect to any translation into spherical coordinates, θ is extremal. The only such points are the corners of the region. At $(a, b, c) = (2^{-1/2}, 2^{-1/2}, 0)$, $X = 2$. However, this maximum is not attained because these corners are not in the feasible region.

Also solved by R. Agnew, A. Alt, R. Bagby, D. Beckwith, H. Caerols & R. Pellicer (Chile), R. Chapman (U.K.), H. Chen, C. Curtis, Y. Dumont (France), D. Fleischman, J.-P. Grivaux (France), E. A. Herman, F. Holland (Ireland), T. Konstantopoulos (U.K.), O. Kouba (Syria), A. Lenskold, J. H. Lindsey II, P. Perfetti (Italy), N. C. Singer, R. Stong, T. Tam, R. Tauraso (Italy), M. Tetiva (Romania), E. I. Verriest, Z. Vörös (Hungary), S. Wagon, G. D. White, GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

Max Min Coordinate Difference

11450 [2009, 647]. *Proposed by Cosmin Pohoata (student), National College "Tudor Vianu," Bucharest, Romania.* Let A be the unit ball in $\mathbb{R}^n$. Find

$$\max_{\mathbf{a}\in A}\left\{\min_{1\le i<j\le n}\left|a_i - a_j\right|\right\}.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let M_n denote the desired maximum. It is implicit in the statement of the problem that $n \ge 2$. We show that $M_n = \sqrt{12/(n(n^2-1))}$.

Let $(a_1, \ldots, a_n)$ be an element of A at which the maximum is achieved, and let $M_n = \min\{|a_i - a_j|\colon 1 \le i < j \le n\}$. There is a permutation σ of the set $\{1, 2, \ldots, n\}$ such that $a_{\sigma(1)} \le a_{\sigma(2)} \le \cdots \le a_{\sigma(n)}$. Write for simplicity $b_j = a_{\sigma(j)}$. For $j > i$, we then have

$$b_j - b_i = \sum_{k=i+1}^{j}(b_k - b_{k-1}) \ge (j-i)M_n.$$

From this we conclude that $|b_j - b_i| \geq |j - i|\, M_n$ for $1 \leq i, j \leq n$. Therefore

$$\begin{aligned} M_n^2 \sum_{1\leq i,j\leq n} (j-i)^2 &\leq \sum_{1\leq i,j\leq n} (b_j - b_i)^2 = \sum_{1\leq i,j\leq n} (a_j - a_i)^2 \\ &= \sum_{1\leq i,j\leq n} (a_j^2 + a_i^2 - 2a_i a_j) \\ &\leq 2n \sum_{k=1}^{n} a_k^2 - 2\left(\sum_{k=1}^{n} a_k\right)^2 \leq 2n, \end{aligned}$$

since $\sum_{k=1}^{n} a_k^2 \leq 1$ when $(a_1, \ldots, a_n) \in A$. On the other hand,

$$\sum_{1\leq i,j\leq n} (j-i)^2 = 2n \sum_{k=1}^{n} k^2 - 2\left(\sum_{k=1}^{n} k\right)^2 = \frac{n^2(n^2-1)}{6}.$$

It follows that $M_n^2 \leq 12/(n(n^2-1))$, so $M_n \leq \sqrt{12/(n(n^2-1))}$.

Conversely, if we consider $(a_1^{(0)}, a_2^{(0)}, \ldots, a_n^{(0)})$ defined by

$$a_k^{(0)} = \sqrt{\frac{12}{n(n^2-1)}}\left(k - \frac{n+1}{2}\right), \qquad k = 1, 2, \ldots, n,$$

then $(a_1^{(0)}, \ldots, a_n^{(0)}) \in A$ and

$$\min_{1\leq i<j\leq n} \left|a_i^{(0)} - a_j^{(0)}\right| = \sqrt{\frac{12}{n(n^2-1)}}.$$

Thus $M_n \geq \sqrt{12/(n(n^2-1))}$.

Editorial comment. Marian Tetiva (Romania) notes that a stronger form of this problem appeared as Problem E2032, this MONTHLY **76** (1969) 691–692, proposed by D. S. Mitrinović. See also Problem 3.9.9 in Mitrinović, *Analytic Inequalities* (Springer-Verlag, 1970).

Also solved by A. Alt, R. F. de Andrade, M. R. Avidon, R. Bagby, D. Beckwith, J. Cade, R. Chapman (U.K.), L. Comerford, W. J. Cowieson, P. P. Dályay (Hungary), A. Fielbaum (Chile), D. Fleischman, O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, A. Ilić (Serbia), T. Konstantopoulos (U.K.), J. Kuplinsky, J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, D. Ray, K. Schilling, B. Schmuland (Canada), J. Simons (U.K.), R. Stong, M. Tetiva (Romania), E. I. Verriest, GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

A Cauchy–Schwarz Puzzle

11458 [2009, 747]. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simon Stoilow" of the Romanian Academy, Bucharest, Romania.* Let $a_1, \ldots, a_n$ be nonnegative and let r be a positive integer. Show that

$$\left(\sum_{1\leq i,j\leq n} \frac{i^r j^r a_i a_j}{i+j-1}\right)^2 \leq \sum_{m=1}^{n} m^{r-1} a_m \sum_{1\leq i,j,k\leq n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.$$

Solution by Francisco Vial, student, Pontificia Universidad Católica de Chile, Santiago, Chile. Let $f(x) := \sum_{i=1}^{n} i^r a_i x^{i-1}$, so

$$\int_0^1 f(x)\,dx = \sum_{m=1}^{n} m^{r-1} a_m,$$

$$\int_0^1 f^2(x)\,dx = \int_0^1 \left(\sum_{1\le i,j\le n} i^r j^r a_i a_j x^{i+j-2} \right) dx = \sum_{1\le i,j\le n} \frac{i^r j^r a_i a_j}{i+j-1}, \qquad \text{and}$$

$$\int_0^1 f^3(x)\,dx = \int_0^1 \left(\sum_{1\le i,j,k\le n} i^r j^r k^r a_i a_j a_k x^{i+j+k-3} \right) dx = \sum_{1\le i,j,k\le n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.$$

The stated inequality is equivalent to

$$\left(\int_0^1 f^2(x)\,dx \right)^2 \le \left(\int_0^1 f(x)\,dx \right) \left(\int_0^1 f^3(x)\,dx \right),$$

which follows by applying the Cauchy–Schwarz inequality to $f(x)^{1/2}$ and $f(x)^{3/2}$.

Remarks. Because $a_1, \ldots, a_n$ are nonnegative, $f(x)$ is nonnegative and continuous on $[0, 1]$, so $f(x)^{1/2}$ and $f(x)^{3/2}$ are real and well defined. The parameter r need not be an integer.

Also solved by M. R. Avidon, R. Chapman (U.K.), P. P. Dályay (Hungary), D. Grinberg, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Simons (U.K.), R. Stong, GCHQ Problem Solving Group (U.K.), and the proposers.

An Orthocenter Inequality

11461 [2009, 844]. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.* Let a, b, and c be the lengths of the sides opposite vertices A, B, and C of an acute triangle. Let H be the orthocenter. Let d_a be the distance from H to side BC, and similarly for d_b and d_c. Show that

$$\frac{1}{d_a + d_b + d_c} \ge \frac{2}{3} \left(\frac{3}{abc} \left(\frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} + \frac{1}{\sqrt{ab}} \right) \right)^{1/4}.$$

Solution by Michael Vowe, Fachhochschule Nordwestschweiz, Muttenz, Switzerland. Let R be the circumradius, r the inradius, F the area, and s the semiperimeter. From $d_a = 2R\cos B\cos C$, $d_b = 2R\cos C\cos A$, $d_c = 2R\cos A\cos B$, we obtain

$$d_a + d_b + d_c = 2R\big(\cos A\cos B + \cos B\cos C + \cos C\cos A\big) \le 2r\left(1 + \frac{r}{R}\right)$$

(see 6.10, p. 181, in D. Mitrinovic et al., *Recent Advances in Geometric Inequalities,* Dordrecht, 1989). From Jensen's inequality for concave functions (here, the square root), we have

$$\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \le 3 \cdot \sqrt{\frac{1}{3}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)} = \sqrt{\frac{6s}{abc}}.$$

From $abc = 4RF = 4Rrs$ and $s^2 \geq 27r^2$ (6.1, p. 180, ibid.) we get

$$\frac{2}{3}\left(\frac{3}{abc}\left(\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}\right)\right)^{1/4} \leq \frac{2}{3}\left(\frac{3\sqrt{6s}}{(abc)^{3/2}}\right)^{1/4}$$

$$\leq \frac{2}{3}\left(\frac{3\sqrt{6}}{(4Rr)^{3/2}}\cdot\frac{1}{3\sqrt{3}\,r}\right)^{1/4} = \frac{2^{3/8}}{3}\cdot\frac{1}{R^{3/8}}\cdot\frac{1}{r^{5/8}}.$$

Thus it suffices to prove

$$\frac{1}{2r\left(1+\dfrac{r}{R}\right)} \geq \frac{2^{3/8}}{3}\cdot\frac{1}{R^{3/8}}\cdot\frac{1}{r^{5/8}}.$$

Writing $x = r/R$, this means we must prove $x^{3/8}(1+x) \leq 3/(2\cdot 2^{3/8})$ for $0 < x \leq 1/2$. The function $f(x) = x^{3/8}(1+x)$ is increasing on $[0, 1/2]$, though, and we are done.

Equality holds only if $x = 1/2$, or equivalently, $R = 2r$, which makes the triangle equilateral.

Also solved by P. P. Dályay (Hungary), O. Faynshteyn (Germany), K.-W. Lau (China), C. R. Pranesachar (India), R. Stong, GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

An Erroneous Claim

11465 [2009, 845]. *Proposed by Pantelimon George Popescu, Polytechnic University of Bucharest, Bucharest, Romania, and José Luis Díaz-Barrero, Polytechnic University of Catalonia, Barcelona, Spain.* Consider three simple closed curves in the plane, of lengths p_1, p_2, and p_3, enclosing areas A_1, A_2, and A_3, respectively. Show that if $p_3 = p_1 + p_2$ and $A_3 = A_1 + A_2$, then $8\pi A_3 \leq p_3^2$.

Solution by the Texas State University Problem Solvers Group, San Marcos, TX. The problem as stated is false. Consider the following counterexample. Let the first curve be a square of side 1, so $p_1 = 4$ and $A_1 = 1$. Let the second curve be a square with $p_2 = 40$ and $A_2 = 100$. Let the third curve be a rectangle with sides $11 + 2\sqrt{5}$ and $101/(11 + 2\sqrt{5})$ so that $p_3 = 44$ and $A_3 = 101$. These three curves fulfill the requirements of the problem, and yet $8\pi A_3 > p_3^2$.

Let us incorporate the additional requirement that $p_1^2 + p_2^2 = 2p_1p_2$. Then the required inequality can be proved as follows. The isoperimetric inequality applied to any of the curves is

$$A_i \leq \pi\left(\frac{p_i}{2\pi}\right)^2,$$

and thus $4\pi A_i \leq p_i^2$. Therefore

$$4\pi A_3 = 4\pi A_1 + 4\pi A_2 \leq p_1^2 + p_2^2.$$

With the newly-added condition we get

$$8\pi A_3 = 8\pi A_1 + 8\pi A_2 \leq 2p_1^2 + 2p_2^2 = p_1^2 + p_2^2 + 2p_1p_2 = (p_1+p_2)^2 = p_3^2.$$

Also solved by G. Apostolopoulos (Greece), R. Bagby, B. Burdick, R. Chapman (U.K.), W. J. Cowieson, P. P. Dályay (Hungary), J.-P. Grivaux (France), K. Hanes, J. H. Lindsey II, M. D. Meyerson, J. Minkus, J. Simons (U.K.), R. Stong, and the Microsoft Research Problems Group.