PROBLEMS AND SOLUTIONS

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PROPOSED PROBLEMS AND SOLUTIONS

11284. Proposed by Greg Oman, Ohio State University, Columbus, OH. Let \( R \) be
an infinite commutative ring with identity. Suppose that every proper ideal of \( R \) has
smaller cardinality than \( R \). Prove that \( R \) is a field.

11285. Proposed by Yakub Aliyev, Qafqaz University and Baku State University, Baku,
Azerbaijan. Let six points be chosen in cyclic order on the sides of triangle \( ABC \): \( A_1 \)
and \( A_2 \) on \( BC \), \( B_1 \) and \( B_2 \) on \( CA \), and \( C_1 \) and \( C_2 \) on \( AB \). Let \( K \) denote the
intersection of \( A_1B_2 \) and \( C_1A_2 \), \( L \) the intersection of \( B_1C_2 \) and \( A_1B_2 \), and \( M \) the intersection
of \( C_1A_2 \) and \( B_1C_2 \). Let \( T \), \( U \), and \( V \) be the intersections of \( A_1B_2 \) and \( B_1A_2 \), \( B_1C_2 \)
and \( B_2C_1 \), and \( C_1A_2 \) and \( C_2A_1 \), respectively. Prove that lines \( AK \), \( BL \), and \( CM \)
are concurrent if and only if points \( T \), \( U \), and \( V \) are collinear.

11286. Proposed by Marc LeBrun, Fixpoint Inc., Larkspur, CA, and David Applegate
and N.J.A. Sloane, AT&T Shannon Labs, Florham Park, NJ. When \( a \) and \( b \) are positive
integers with \( b \geq 10 \), write \( a_b \) (or \( a : b \) inline) for the integer whose base \( b \) expansion
is the decimal expansion of \( a \). That is, if \( a = \sum_{i=0}^{k} a_i 10^i \) with each \( a_i \) in \( \{0, 1, \ldots , 9\} \),
then \( a_b = a : b = \sum_{i=0}^{k} a_i b^i \). Thus,

\[
10_{11_{12_{13}}} = 10 : (11 : (12 : 13)) = 16.
\]

Consider the “dungeon sequences”

\[
10, 10 : 11, (10 : 11) : 12, ((10 : 11) : 12) : 13 \ldots ,
10, 10 : 11, 10 : (11 : 12), 10 : (11 : (12 : 13)) \ldots ,
10, 11 : 10, (12 : 11) : 10, ((13 : 12) : 11) : 10 \ldots ,
10, 11 : 10, 12 : (11 : 10), 13 : (12 : (11 : 10)) \ldots .
\]

Let \( s_n \) be the \( n \)th term in any of these sequences. Show that \( \log \log s_n/(n \log \log n) \)
approaches 1 as \( n \) goes to infinity.
11287. Proposed by Stephen J. Herschkorn, Highland Park, NJ. Players 1 through
n play “continuous blackjack.” At his turn, Player k considers a random number
Xk drawn from the uniform distribution on [0, 1]. He may either accept Xk as his score
or draw a second number Yk from the same distribution, in which case his score is
Xk + Yk if Xk + Yk < 1 and 0 otherwise. The highest score wins. Give a rule for when
player k should draw a second number, in terms of k, n, the result of Xk, and the
highest score attained so far.

11288. Proposed by Christopher Hillar, Texas A&M University, College Station, TX
and Troels Windfeldt, University of Copenhagen, Copenhagen, Denmark. Let n be a
positive integer, and let U = {1, ..., 2n}. For a set S ⊆ U and a positive integer d, let
h^d_S be the sum of all monomials of degree d in the indeterminates {X_i : X_i ∈ S}. Let T
be the set of all n-element subsets of U with the property that for any odd element k of
the set, k + 1 is not a member. For S in T, let o(S) denote the number of odd elements
of S. Show that for every positive integer d,

\[ h^d_T \prod_{i=1}^{n} (X_{2i-1} - X_{2i}) = \sum_{S \in T} (-1)^{o(S)} h^{d+n}_U \setminus S. \]

11289. Proposed by Oleh Faynshteyn, Leipzig, Germany. Let ABC be a triangle with
sides a, b, and c, all different, and corresponding angles α, β, and γ. Show that
(a) (a + b) \cot(\beta + \frac{1}{2}\gamma) + (b + c) \cot(\gamma + \frac{1}{2}\alpha) + (a + c) \cot(\alpha + \frac{1}{2}\beta) = 0.
(b) (a - b) \tan(\beta + \frac{1}{2}\gamma) + (b - c) \tan(\gamma + \frac{1}{2}\alpha) + (c - a) \tan(\alpha + \frac{1}{2}\beta) = 4(R + r),
where R is the circumradius of the triangle and r the inradius.

11290. Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romani-
a, and Tudorel Lupu, Decibal Highschool, Constanza, Romania. Let f and g be
continuous real valued functions on [0, 1]. Prove that there exists c in (0, 1) such that

\[ \int_{x=0}^{1} f(x) \, dx \int_{x=0}^{c} xg(x) \, dx = \int_{x=0}^{1} g(x) \, dx \int_{x=0}^{c} xf(x) \, dx. \]

SOLUTIONS

Choice Bounds

11132 [2005, 180]. Proposed by Jonathan Sondow, New York, NY. Let m be a positive
integer and let r be a real number with r ≥ 1. Prove that

\[ \frac{1}{4rm} \left( \frac{(r + 1)^{r+1}}{r^r} \right)^m < \left( \frac{(r + 1)^m}{m} \right) < \left( \frac{(r + 1)^{r+1}}{r^r} \right)^m. \]

(When z is real, \( \left( \frac{z}{m} \right) \) denotes \( \frac{1}{m!} \prod_{k=0}^{m-1} (z - k) \).

Solution by Richard Stong, Rice University, Houston, TX. We prove stronger bounds,
replacing 4rm with e\sqrt{m} on the left and multiplying the upper bound by \sqrt{1 + 1/r/e}.

Suppose 1 ≤ a < b and b − a is an integer. Since \log x is increasing and concave
downward on [a, b], comparing the integral to a Riemann sum and to a trapezoidal
approximation gives

\[ \sum_{s=a+1}^{b} \log s - \frac{1}{2} \log \frac{b}{a} < \int_{a}^{b} \log x \, dx = b \log b - a \log a - b + a < \sum_{s=a+1}^{b} \log s. \]
Rearranging and exponentiating yields
\[ e^{a-b} b^a a^{-a} < (a+1)(a+2) \cdots b < e^{a-b} b^a a^{-a} \sqrt{\frac{b}{a}}. \]

Applying these inequalities to the numerator and denominator of \( \frac{(z-m+1) \cdots z}{z-m} \), for \( z \geq m+1 \) we have
\[ \frac{1}{e^{\sqrt{m}}} \left( \frac{z^r}{z-m} \right)^{z-m} m^m < \left( \frac{z}{m} \right) < \frac{1}{e^{\sqrt{m}}} \left( \frac{z^r}{z-m} \right)^{z-m} m^m. \]

For the given problem, setting \( z = (r+1)m \) yields
\[ \frac{1}{e^{\sqrt{m}}} \left( \frac{(r+1)^{r+1}}{r^r} \right)^m < \left( \frac{(r+1)m}{m} \right) < \frac{1}{e^{\sqrt{m}}} \left( \frac{(r+1)^{r+1}}{r^r} \right)^m. \]

Editorial comment. The GCHQ Problem Solving Group (U. K.) proved an upper bound that is stronger still when \( m > 1 \):
\[ \left( \frac{(r+1)m}{m} \right) < e^{-1/3} \sqrt{\frac{r+1}{r}} \frac{1}{\sqrt{m}} \left( \frac{(r+1)^{r+1}}{r^r} \right)^m. \]

Also solved by S. Amghibech (Canada), T. Andebrhan (Eritrea), M. Avidon, M. Bataille (France), D. Beckwith, P. Bracken & N. Nadeau, R. Chapman (U. K.), Y. Dumont (France), G. C. Greubel, O. P. Lossers (Netherlands), M. H. Mehrabi (Iran), M. A. Prasad (India), R. Richberg (Germany), H.-J. Seiffert (Germany), A. Stadler (Switzerland), J. Vinuesa (Spain), A. Vörös (Hungary), L. Zhou, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

An Integral Inequality

11133 [2005, 180]. Proposed by Paul Bracken, University of Texas Pan-American, Edinburg, TX. Let \( f \) be a nonnegative, continuous, concave function on \([0, 1]\) with \( f(0) = 1 \). Prove that
\[ 2 \int_0^1 x^2 f(x) \, dx + \frac{1}{12} \leq \left( \int_0^1 f(x) \, dx \right)^2. \]

Solution by Heinz-Jürgen Seiffert, Berlin, Germany. We prove a more general inequality without the condition that \( f \) is nonnegative on \([0, 1]\). Since \( f \) is concave on \([0, 1]\) with \( f(0) = 1 \), we have, for \( 0 \leq x \leq 1 \) and for arbitrary \( p > 0 \),
\[ xf (x^{1/p}) + 1 - x \leq f \left( x \cdot x^{1/p} + (1-x) \cdot 0 \right) = f \left( x^{1+1/p} \right). \]

Multiplying both sides by \((1+1/p)x^{1/p}\) and then integrating using the substitution \( x = u^p \) on the left side and the substitution \( x = u^{p/(p+1)} \) on the right side, one obtains
\[ (p+1) \int_0^1 x f(x) \, dx + \frac{p}{2p+1} \leq \int_0^1 f(x) \, dx. \]

From \( \int_0^1 f(x) \, dx - \left( \int_0^1 f(x) \, dx \right)^2 \leq \frac{1}{4} \) we conclude that
\[ (p+1) \int_0^1 x f(x) \, dx + \frac{2p-1}{8p+4} \leq \left( \int_0^1 f(x) \, dx \right)^2. \]
The proposed inequality is the particular case $p = 1$.

Editorial comment. Several solvers noted that equality holds if and only if $f(x) = 1 - x$.

Also solved by S. Amghibech (Canada), M. Bataille (France), K. Jusec & W. Troy, J. H. Lindsey II, O. P. Lossers (Netherlands), M. H. Mehrabi (Iran), L. E. Miller, M. A. Prasad (India), R. Richberg (Germany), J. Spellmann, A. Stadler (Switzerland), M. K. Uzun (Turkey), R. Tauraso (Italy), R. M. Torrejon, Z. Zhou, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

A Combinatorial Maximum

11142 [2005, 273]. Proposed by Donald Knuth, Stanford University, Stanford, CA. Let $\binom{n}{m}$ be the (absolute value of) the $(n, m)$th Stirling number of the first kind, namely the number of permutations of $n$ objects having $m$ cycles. Given that $n$ is a positive integer and that $\binom{n}{m}^m = \max_{1 \leq k \leq n} \binom{n}{k} n^k$, prove that

$$n \log 2 - \frac{3}{4} < m < n \log 2 + \frac{4}{3}.$$

Solution by Roberto Tauraso, Università di Roma “Tor Vergata”, Rome, Italy. We will use the following result due to J. N. Darroch, “On the distribution of the number of successes in independent trials,” Ann. Math. Statist. 35 (1964), 1317–1321: Let $P(z) = \sum_{k=0}^{n} a_k z^k$. If $n \geq 1$, all $a_k \geq 0$, and all roots of $P$ are real, then the sequence of coefficients is unimodal and attains its maximum for at most two indices. Furthermore, indices attaining the maximum differ from $P'(1)/P(1)$ by at most 1.

In particular, let $P(x) = \prod_{i=1}^{n} (nx + i - 1)$. It is well known that the generating function for permutations on $n$ elements, indexed by the number of cycles, is the rising factorial; that is, $\sum_{k=0}^{n} \binom{n}{k} x^k = \prod_{i=1}^{n} (x + i - 1)$. Thus

$$P(x) = \prod_{i=1}^{n} (nx + i - 1) = \sum_{k=0}^{n} \binom{n}{k} n^k x^k.$$

The coefficients are nonnegative and all roots are real. By Darroch’s Theorem, the largest coefficient $a_m$ occurs for $m$ within 1 of $P'(1)/P(1)$. Letting $\mu = P'(1)/P(1)$, this means that $\mu - 1 < m < \mu + 1$. We compute

$$\mu = \frac{P'(1)}{P(1)} = n \left( \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} \right) = n(H_{2n} - H_n) + \frac{1}{2},$$

where $H_n$ is the harmonic number $\sum_{i=1}^{n} 1/i$. It is known that for $n \geq 1$, there exists $\epsilon_n$ strictly between 0 and 1 such that

$$H_n = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\epsilon_n}{120n^4}.$$

(See R. L. Graham D. E. Knuth, and O. Patashnik, Concrete Mathematics, 2nd edition, Addison-Wesley, 1994, p. 480.) Thus

$$\mu = n \log 2 + \frac{1}{4} + \frac{1}{16n} - \frac{\epsilon_n}{120n^3} + \frac{\epsilon_{2n}}{1920n^3}.$$

Finally,

$$\mu + 1 < n \log 2 + \frac{5}{4} + \frac{1}{16n} + \frac{1}{1920n^3} \leq n \log 2 + \frac{5}{4} + \frac{1}{16} + \frac{1}{1920} < n \log 2 + \frac{4}{3}.$$

April 2007] PROBLEMS AND SOLUTIONS 361
and
\[ \mu - 1 > n \log 2 - \frac{3}{4} + \frac{1}{16n} - \frac{1}{120n^3} > n \log 2 - \frac{3}{4}. \]

Also solved by Allen Stenger and the proposer.

Inverting a Matrix of Stirling Numbers

11156 [2005, 467]. Proposed by Richard Stanley, Massachusetts Institute of Technology, Cambridge, MA. Let \( s(n, k) \) denote a signed Stirling number of the first kind. It is well known that the matrix \([s(n, k)] (n, k \geq 1)\) has inverse \([S(n, k)] (n, k \geq 1)\), where \( S(n, k) \) denotes a Stirling number of the second kind. Find a combinatorial formula for the inverse of the finite matrix \([s(n, n-k+1)] \) \((1 \leq n, k \leq N)\), where we set \( s(n, j) = 0 \) if \( j \leq 0 \). For example, when \( N = 4 \) this formula should give the numbers obtained by inverting

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & -3 & 2 & 0 \\
1 & -6 & 11 & -6
\end{pmatrix}.
\]

**Solution by GCHQ Problem Solving Group, Cheltenham, U. K.** Let \( A \) be the matrix with entry \( a_{i,j} = s(i, i-k+1) \). We show that \( A^{-1} = B \), where \( B \) has entries

\[
b_{i,j} = (-1)^{j-1} j \frac{1}{(j-1)!} \sum_{r=1}^{j} (-1)^{r-1} \binom{j-1}{r-1} \frac{1}{r^{i+1-j}}.
\]

Let \( C = AB \). Since \( a_{i,k} = 0 \) if \( k > i \), we have

\[
c_{i,j} = \sum_{k=1}^{i} a_{i,k} b_{k,j} = \sum_{k=1}^{i} s(i, i-k+1) b_{k,j} = \sum_{m=1}^{i} s(i, m) b_{i+1-m, j}
\]

\[
= (-1)^{j-1} \frac{j}{(j-1)!} \sum_{m=1}^{i} s(i, m) \sum_{r=1}^{j} (-1)^{r-1} \binom{j-1}{r-1} \frac{1}{r^{i+2-m-j}}
\]

\[
= (-1)^{j-1} \frac{j}{(j-1)!} \sum_{r=1}^{j} (-1)^{r-1} \binom{j-1}{r-1} r^{j-2+i} \sum_{m=1}^{i} s(i, m) r^m.
\]

It is well known that \( \sum_{m=1}^{j} s(i, m) r^m = r(r-1)(r-2) \cdots (r-i+1) \). When \( j > i \), it follows that \( r^{j-2+i} \sum_{m=1}^{j} s(i, m) r^m \) is a polynomial in \( r \) of degree \( j - 2 \); hence its \((j-1)\)st difference is 0. This yields \( c_{i,j} = 0 \) for \( j > i \).

Also, the value of \( r(r-1)(r-2) \cdots (r-i+1) \) is 0 for \( r \in \{1, \ldots, i-1\} \), so when \( j < i \) each summand of the outer sum in \( c_{i,j} \) is 0, and again \( c_{i,j} = 0 \).

When \( j = i \), the only nonzero contribution is when \( r = j \), and

\[
c_{j,j} = (-1)^{j-1} \frac{j}{(j-1)!} (-1)^{j-1} j^{-2} j! = 1.
\]

Hence \( C \) is the identity and \( B = A^{-1} \).

Also solved by D. Callan, R. Chapman (U. K.), Microsoft Research Problems Group, and the proposer.
Compositions Without Common Factors

11161 [2005, 567]. Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY. Show that for all integers \( n \geq 3 \) the number of compositions of \( n \) into relatively prime parts is a multiple of 3. (A composition of \( n \) into \( k \) parts is a list of \( k \) positive integers that sum to \( n \). Thus, there are six compositions of 4 into relatively prime parts: \((3, 1)\), \((1, 3)\), \((2, 1, 1)\), \((1, 2, 1)\), \((1, 1, 2)\), and \((1, 1, 1, 1)\).)

**Solution by the Lafayette College Problem Group, Lafayette College, Easton, PA.** Let \( f(n) \) be the number of compositions of \( n \) into relatively prime parts, with the convention that \( (n) \) is a composition into “relatively prime parts” if and only if \( n = 1 \).

Note that \( f(1) = f(2) = 1 \) and that \( f(3) = 3 \). The number of unrestricted compositions of \( n \) is \( 2^{n-1} \). (In a string of \( n \) dots, insert dividing bars at any subset of the \( n - 1 \) places between two 1’s.) For \( n \geq 3 \),

\[
f(n) = 2^{n-1} - \sum_{d|n, d > 1} f\left(\frac{n}{d}\right)
\]

where the second term subtracts off the compositions into parts with a common divisor, grouped by the greatest common divisor, \( d \).

We show that 3 divides \( f(n) \) by induction on \( n \). For \( n > 3 \), note that \( 2^{n-1} \equiv 2 \pmod{3} \) when \( n \) is even and \( 2^{n-1} \equiv 1 \pmod{3} \) when \( n \) is odd. The induction hypothesis yields \( f(n) \equiv 2^{n-1} - f(1) - f(2) \pmod{3} \) when \( n \) is even and \( f(n) \equiv 2^{n-1} - f(1) \pmod{3} \) when \( n \) is odd, so we conclude that always \( f(n) \equiv 0 \pmod{3} \).

**Editorial comment.** Several solvers used Möbius inversion to derive the explicit formula \( f(n) = \sum_{d|n} \mu(n/d)2^{d-1} \), where \( \mu \) is the Möbius function. A corollary, noted by Tim Keller, is that \( f(n) \) is odd if and only if \( n \) is squarefree.


Counting Lattice Quadrilaterals Tangent to an Ellipse

11163 [2005, 567]. Proposed by Michel Bataille, Rouen, France. Let \( c \) and \( n \) be positive integers with \( n > c^2 \). Let \( q_{n,c} \) denote the number of quadrilaterals with vertices at integer lattice points and sides tangent to the ellipse with equation \( x^2/n + y^2/(n-c^2) = 1 \).

(a) For which \( c \) and \( n \) is \( q_{n,c} \) positive?

(b) Show that, for \( c \geq 1 \), \( \sup_{n > c^2} q_{n,c} = \infty \).

(c)* Is \( q_{n,c} \) finite for all \( n \) and \( c \)?

**Solution by Richard Stong, Rice University, Houston, TX.**

(a) We show that \( q_{n,c} > 0 \) if and only if \( n \) is a sum of two squares.

By substituting

\[
y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1
\]

into the equation for the ellipse and setting the discriminant of the resulting quadratic equation equal to 0, one can establish that the line through \((x_1, y_1)\) and \((x_2, y_2)\) is tangent to the ellipse if and only if

\[
(x_1y_2 - y_1x_2)^2 = (n - c^2)(x_2 - x_1)^2 + n(y_2 - y_1)^2.
\]
Assuming that $q_{n,c} > 0$, let $(x_1, y_1)$ and $(x_2, y_2)$ be two adjacent vertices of a quadrilateral with the desired properties. Rewriting the previous equation as

$$(x_1 y_2 - y_1 x_2)^2 + (c x_2 - c x_1)^2 = n[(x_2 - x_1)^2 + (y_2 - y_1)^2],$$

we see that the left side and the multiplier of $n$ on the right side are both sums of two squares. It follows that $n$ is also a sum of two squares, since a positive integer is a sum of two squares if and only if all primes congruent to 3 modulo 4 occur with even exponent in its prime factorization (see, for example, Niven, Zuckerman, and Montgomery, *An Introduction to the Theory of Numbers*, 5th edition, p. 55).

Conversely, suppose that $n = a^2 + b^2$, where $a$ and $b$ are integers. Consider the four vertices $(a - b, a + b - c)$, $(a + b, b + c - a)$, $(b - a, -a - b - c)$, and $(-a - b, a + c - b)$. The lines through consecutive vertices are tangent to the ellipse, by the formula above, so $q_{n,c} > 0$.

(b) When $n = a^2 + b^2$, the midpoints of the sides of the quadrilateral given above are $(a, b)$, $(b, -a)$, $(-a, -b)$, and $(-a, b)$. Hence the quadrilaterals built from different expressions of $n$ as a sum of two squares are distinct. Since the number of ways that an integer can be expressed as a sum of two squares is unbounded (see, for example, Section 3.6 of Niven, Zuckerman, and Montgomery), also $q_{n,c}$ is unbounded for any $c$.

(c) For a point $P$ in the plane, let $\theta(P)$ be the angle subtended by the ellipse from the point $P$. The function $\theta$ is defined and continuous for $P$ outside the ellipse. Also $\theta(P) < \pi$ and $\theta(P) \to 0$ as $P$ tends to infinity. In particular, the maximum of $\theta(P)$ over lattice points outside the ellipse is attained. Let the maximum be $\pi - \epsilon$. If the angles of a quadrilateral with the desired properties are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, indexed in non-increasing order, then

$$2\pi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 2(\pi - \epsilon) + 2\alpha_3,$$

and hence $\alpha_3 \geq \epsilon$. It follows that there is a compact set that contains at least three of the vertices of every such quadrilateral. Thus there are only finitely many possibilities for three of the vertices. Three vertices determine the quadrilateral, however, so the number of such quadrilaterals is finite.

Part (c) also solved by GCHQ Problem Solving Group (U. K.). Parts (a) and (b) also solved by the proposer.

**A Recurrent Identity**

11164 [2005, 568]. Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. Show that if $n$ is a positive integer, then

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sum_{1 \leq i \leq j \leq k} \frac{1}{ij} = \frac{1}{n^2}.$$

Solution I by Ulrich Abel, University of Applied Sciences Giessen-Friedberg, Giessen and Friedberg, Germany, and Mircea Ivan, Technical University of Cluj-Napoca, Romania. Let $S_n$ be the left side of the desired equality above. Interchanging the order of summation, we obtain

$$S_n = \sum_{i=1}^{n} \frac{1}{i} \sum_{j=i}^{n} \frac{1}{j} \sum_{k=j}^{n} (-1)^{k+1} \binom{n}{k}.$$

It is well known that

$$\sum_{k=j}^{n} (-1)^{k} \binom{n}{k} = \sum_{k=j}^{n} (-1)^{k} \left[ \binom{n-1}{k-1} + \binom{n-1}{k} \right] = (-1)^{j} \binom{n-1}{j-1},$$
and hence \( \sum_{k=j}^{n} (-1)^{k+1} \binom{n}{k} = (-1)^{j+1} \frac{j}{n} \binom{n}{j} \). Using this identity twice yields

\[
S_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{i} \sum_{j=i}^{n} (-1)^{j+1} \binom{n}{j} = \frac{1}{n^2} \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} = \frac{1}{n^2}.
\]

Solution II by Yong Duk Kim, Seoul National University; Hwa-Young Lee, Jangpyung Middle School; Yong Hah Lee, Ewha Womans University; and Kye Hee Park, Jangan Middle School, Seoul, Korea. Define

\[
f_{n}(x, y) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sum_{j=1}^{k} \sum_{i=1}^{j} x^{i-1} y^{j-1}.
\]

Note that

\[
\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{1-z^k}{1-z} = (1-z)^{n-1}.
\]

With two geometric sums and two invocations of the identity above, we have

\[
f_{n}(x, y) = \frac{1}{1-x} \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \left[ \frac{1-y^k}{1-y} - \frac{x(1-(xy)^k)}{1-xy} \right]
\]

\[= \frac{1}{1-x} \left[(1-y)^{n-1} - x(1-xy)^{n-1}\right].
\]

Integrating the defining and final expressions for \( f_{n}(x, y) \) on \([0, 1] \times [0, 1] \) yields

\[
\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sum_{1 \leq i \leq j \leq k} \frac{1}{ij} = \int_{0}^{1} \frac{1}{1-x} \int_{0}^{1} [(1-y)^{n-1} - x(1-xy)^{n-1}] dy dx
\]

\[= \frac{1}{n} \int_{0}^{1} \frac{1}{1-x} [1 + (1-x)^{n} - 1] dx = \frac{1}{n^2}.
\]

Editorial comment. It seems to have gone unnoticed by solvers and editors alike that the requested identity was proved in Solution II of Problem 10490 in this Monthly, [1995, 930; 1997, 588–590]. Several solvers here proved the more general result below, which also follows from the techniques in that solution and the techniques above.

\[
\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \sum_{1 \leq i_{1} \leq \cdots \leq i_{m} \leq k} \frac{1}{i_{1} \cdots i_{m}} = \frac{1}{n^{m}}.
\]


April 2007] PROBLEMS AND SOLUTIONS 365