REFINEMENTS OF THE FINSLER-HADWIGER REVERSE INEQUALITY

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Abstract. In this note we give refinements of the celebrated Finsler-Hadwiger reverse inequality which states that in any triangle \( ABC \) with sides of lengths \( a, b, c \) the following inequality is valid
\[
a^2 + b^2 + c^2 \leq 4S\sqrt{3} + k[(a - b)^2 + (b - c)^2 + (c - a)^2],
\]
where \( S \) denotes the area of the triangle \( ABC \) for \( k = 3 \). We refine the above inequality by proving that it remains true for \( k = 2 \) and we also show that this new improvement fails in the case when the triangle \( ABC \) is not acute angled. In the final part of this work, we improve the Finsler-Hadwiger reverse inequality for \( k = \frac{6 - \sqrt{6}}{2} \) and we conjecture that the same result holds for \( k = \frac{2 - \sqrt{3}}{\sqrt{3} - 2} \).

1. Introduction & Main results

Many of the most important results in theory of geometric inequalities were discovered by the beginning of the 20-th century. The first important inequality was discovered in 1919 and it is due to Weitzenbock (see [3]), namely

\[
\text{Theorem 1.1. In any triangle } ABC \text{ with sides of lengths } a, b, c \text{ the following inequality holds}
\]
\[
a^2 + b^2 + c^2 \geq 4S\sqrt{3},
\]
where \( S \) denotes the area of the triangle \( ABC \).

The above theorem also appeared in International Mathematical Olympiad in 1961 and many proofs of it can be found in [2]. A refinement of Theorem 1.1 is the Finsler-Hadwiger inequality

\[
\text{Theorem 1.2. In any triangle } ABC \text{ with sides of lengths } a, b, c \text{ the following inequality holds}
\]
\[
a^2 + b^2 + c^2 \geq 4S\sqrt{3} + (a - b)^2 + (b - c)^2 + (c - a)^2,
\]
where \( S \) denotes the area of the triangle \( ABC \).

Many proofs of Theorem 1.2 can be found in [2] as well as other new proofs in [8], [6] or [10]. The reverse of Theorem 1.2 to which we give two different proofs from [11] in what follows, states that

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\]
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Theorem 1.3. In any triangle $ABC$ with sides of lengths $a, b, c$ the following inequality holds

$$a^2 + b^2 + c^2 \leq 4S\sqrt{3} + 3[(a - b)^2 + (b - c)^2 + (c - a)^2],$$

where $S$ denotes the area of the triangle $ABC$.

*First proof.* Let us denote by $a = y + z, b = z + x$ and $c = x + y$ where $x, y, z > 0$. The area is given by the formula $S = \sqrt{xyz(x + y + z)}$. Our inequality becomes

$$(x + y)^2 + (y + z)^2 + (z + x)^2 \leq 4\sqrt{3}xyz(x + y + z) + 3\sum_{cyc}(x - y)^2$$

which is equivalent after a few calculations with

$$2\sum_{cyc}xy - \sum_{cyc}x^2 \leq \sqrt{3}xyz(x + y + z).$$

But, from Schur’s inequality (see [7]), i.e.

$$(\sum_{cyc}x^3) + 3xyz \geq \sum_{cyc}xy(x + y)$$

one can easily deduce that

$$2\sum_{cyc}xy - \sum_{cyc}x^2 \leq \frac{9xyz}{x + y + z}.$$

Now, we are only left to prove that $(x + y + z)^3 \geq 27xyz$, which follows immediately from AM-GM inequality.

*Second proof.* The inequality can be rewritten

$$6(ab + bc + ca) \leq 4S\sqrt{3} + 5(a^2 + b^2 + c^2).$$

Since $ab + bc + ca = s^2 + 4Rr + r^2$ and $a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2)$, our inequality is successively equivalent to

$$6(s^2 + 4Rr + r^2) \leq 4S\sqrt{3} + 10(s^2 - 4Rr - r^2),$$

$$64Rr + 16r^2 \leq 4S\sqrt{3} + 4s^2,$$

$$s^2 + S\sqrt{3} \geq 16Rr + 4r^2.$$ 

By applying Gerretsen’s inequality, $s^2 \geq 16Rr - 5r^2$, we have $s^2 + S\sqrt{3} \geq 16Rr - 5r^2 + S\sqrt{3}$. Thus it is enough to show that

$$16Rr - 5r^2 + S\sqrt{3} \geq 16Rr + 4r^2$$

which reduces to the well-known Mitrinovic inequality $s^2 \geq 3\sqrt{3}S$. $\square$

We shall prove that in the case of an acute-angled triangle Theorem 1.3 can be improved in the following sense
Theorem 1.4. In any acute-angled triangle $ABC$ with sides of lengths $a, b, c$, the following inequality holds

$$a^2 + b^2 + c^2 \leq 4S\sqrt{3} + 2[(a - b)^2 + (b - c)^2 + (c - a)^2],$$

where $S$ denotes the area of the triangle $ABC$.

In order to prove our main results we shall use the following lemmas due to Popoviciu (see [5]) and Walker (see [9]), i.e.

Lemma 1.5. Let $I$ be an interval and $f : I \rightarrow \mathbb{R}$ be a convex function. For any $x, y, z \in I$ the following inequality holds:

$$f(x) + f(y) + f(z) + f\left(\frac{x + y + z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x + y}{2}\right) + f\left(\frac{y + z}{2}\right) + f\left(\frac{z + x}{2}\right)\right].$$

Lemma 1.6. In any acute triangle $ABC$ with sides of lengths $a, b, c$ and with circumradius $R$ and inradius $r$ the following inequality holds:

$$a^2 + b^2 + c^2 \geq 4(R + r)^2.$$

First proof of Theorem 1.4. By expanding, the given inequality is equivalent to

$$a^2 + b^2 + c^2 \leq 4S\sqrt{3} + 2[2(a^2 + b^2 + c^2) - 2(ab + bc + ca)]$$

which is equivalent to

$$3(a^2 + b^2 + c^2) + 4S\sqrt{3} \geq 4(ab + bc + ca)$$

or

$$4(ab + bc + ca) - 2(a^2 + b^2 + c^2) \leq a^2 + b^2 + c^2 + 4S\sqrt{3}.(*)$$

On the other hand, by the cosine law, we have

$$a^2 = b^2 + c^2 - 2bc \cos A = (b - c)^2 + 2bc(1 - \cos A) = (b - c)^2 + 4S\frac{1 - \cos A}{\sin A} =$$

$$= (b - c)^2 + 4S \tan \frac{A}{2},$$

where we used the well-known formulae $1 - \cos \alpha = 2\sin^2 \frac{\alpha}{2}$ and $\sin \alpha = 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$.

Thus, we obtain

$$a^2 + b^2 + c^2 = (a - b)^2 + (b - c)^2 + (c - a)^2 + 4S\left(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}\right)$$

which is equivalent to

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) = 4S\left(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}\right).(**)$$
Now, by (\(\ast\)) and (\(\ast\ast\)), we have
\[
8S \left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right) \leq a^2 + b^2 + c^2 + 4S\sqrt{3}
\]
which is equivalent to
\[
2 \left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right) \leq \frac{a^2 + b^2 + c^2}{4S} + \sqrt{3}.
\]
Again, by the cosine and sine law, we have
\[
\cot A + \cot B + \cot C = \frac{R(b^2 + c^2 - a^2) + R(c^2 + a^2 - b^2) + R(a^2 + b^2 - c^2)}{abc},
\]
where \(R\) denotes the circumradius of the triangle \(ABC\). Since \(R = \frac{abc}{4S}\), we get
\[
\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4S}.
\]
Now, our inequality finally reduces to
\[
\cot A + \cot B + \cot C + \sqrt{3} \geq 2 \left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right).
\]
Let us consider the function \(f(x) = \cot x\) where \(f : (0, \frac{\pi}{2}) \to \mathbb{R}\). A simple calculation of the second derivative shows that \(f\) is convex. By applying Lemma 1.5 for \(f(x) = \cot x\), we obtain
\[
\frac{1}{3}(\cot A + \cot B + \cot C) + \cot \left( \frac{A + B + C}{3} \right) \geq \frac{2}{3} \sum_{\text{cyc}} \cot \left( \frac{A + B}{2} \right)
\]
which is equivalent to
\[
\cot A + \cot B + \cot C + \sqrt{3} \geq 2 \left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right),
\]
exactly what we wanted to prove.

**Second proof of Theorem 1.4.** The conclusion rewrites as
\[
4(a^2 + b^2 + c^2) - 4(ab + bc + ca) \geq a^2 + b^2 + c^2 - 4S\sqrt{3}
\]
which is equivalent to
\[
3(a^2 + b^2 + c^2) + 4S\sqrt{3} \geq 4(ab + bc + ca).
\]
Since \(ab + bc + ca = s^2 + 4Rr + r^2\) and \(a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2)\), our inequality is equivalent to
\[
6(s^2 - 4Rr - r^2) + 4S\sqrt{3} \geq 4(s^2 + 4Rr + r^2)
\]
which finally reduces to
\[
s^2 + 2S\sqrt{3} \geq 20Rr + 5r^2,
\]
where \(s\) is the semiperimeter of the triangle \(ABC\). By Gerretsen’s inequality, \(s^2 \geq 16Rr - 5r^2\), we have \(s^2 + 2S\sqrt{3} \geq 16Rr - 5r^2 + 2S\sqrt{3}\). Now, we are left to prove
\[
16Rr - 5r^2 + 2S\sqrt{3} \geq 20Rr + 5r^2
\]
which is successively equivalent to

\[ S\sqrt{3} \geq 2Rr + 5r^2, \]
\[ s\sqrt{3} \geq 2R + 5r \]

and by squaring we have

\[ 3s^2 \geq 4R^2 + 20Rr + 25r^2. \]

Applying the Lemma 1.6 written in the form

\[ s^2 \geq 4R^2 + 8Rr + 3r^2, \]

we deduce

\[ 3s^2 \geq 6R^2 + 24Rr + 25r^2. \]

In this moment, it is enough to prove that

\[ 6R^2 + 24Rr + 9r^2 \geq 4R^2 + 20Rr + 25r^2 \]

which is nothing else that \((R - 2r)(R + 4r) \geq 0\). This is evident by Euler’s inequality. \(\square\)

**Remark.** The inequality does not hold in any obtuse triangle due to the following counterexample:

Let there be an isosceles triangle \(ABC\) with \(AB = AC = x\) and \(BC = 1\). We consider it to be obtuse in \(A\) so \(AB^2 + AC^2 < BC^2\) and thus \(x < \frac{1}{\sqrt{2}}\). Also to exist such a triangle \(x\) must fulfill \(x > \frac{1}{2}\).

For the inequality to hold we must have that \(\forall x \in \left(\frac{1}{2}; \frac{1}{\sqrt{2}}\right), 2x^2 + 1 \leq 4S\sqrt{3} + 4(x - 1)^2\). In our case it is easy to see that \(4S\sqrt{3} = \sqrt{3}(4x^2 - 1)\) and thus the inequality is equivalent to \(4x \leq 1 + \sqrt{3}(4x^2 - 1)\). It is obvious that for \(x = 0.55\) the inequality fails and this is a counterexample.

Thus, we think that the constant \(k = 2\) is not optimal. An improvement of Theorem 1.4 is

**Theorem 1.7.** In any acute-angled triangle \(ABC\) with sides of lengths \(a, b, c\) the following inequality holds

\[ a^2 + b^2 + c^2 \leq 4S\sqrt{3} + \frac{(6 - \sqrt{6})}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2], \]

where \(S\) denotes the area of the triangle \(ABC\).

**Proof.** The inequality is equivalent to

\[ 2(a^2 + b^2 + c^2) \leq 8S\sqrt{3} + (6 - \sqrt{6})[2(a^2 + b^2 + c^2) - 2(ab + bc + ca)]. \]

Using the identities \(ab + bc + ca = s^2 + 4Rr + r^2\) and \(a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2)\) our inequality becomes

\[ s^2(4 - \sqrt{6}) + 4S\sqrt{3} \geq Rr(64 - 12\sqrt{6}) + r^2(16 - 3\sqrt{6}). \]

Again, by Gerretsen’s inequality, \(s^2 \geq 16Rr - 5r^2\), we infer that

\[ s^2(4 - \sqrt{6}) + 4S\sqrt{3} \geq (16Rr - 5r^2)(4 - \sqrt{6}) + 4S\sqrt{3}. \]
Now, we are only left to prove that 
\[
(16Rr - 5r^2)(4 \sqrt{6}) + 4S\sqrt{3} \geq Rr(64 - 12\sqrt{6}) + r^2(16 - 3\sqrt{6}),
\]
which can be successively rewritten as
\[
4S\sqrt{3} \geq (64 - 12\sqrt{6} - 64 + 16\sqrt{6})Rr + (16 - 3\sqrt{6} + 20 - 5\sqrt{6})r^2,
\]
\[
4S\sqrt{3} \geq 4\sqrt{6}Rr + (36 - 8\sqrt{6})r^2,
\]
\[
3s^2 \geq 6R^2 + 2(9\sqrt{6} - 12)Rr + (9 - 2\sqrt{6})^2r^2,
\]
\[
3s^2 \geq 6R^2 + 2(9\sqrt{6} - 12)Rr + (105 - 36\sqrt{6})r^2.
\]
By Lemma 1.6 we have \(3s^2 \geq 6R^2 + 24Rr + 9r^2\) and we only need to see that
\[
6R^2 + 24Rr + 9r^2 \geq 6R^2 + 2(9\sqrt{6} - 12)Rr + (105 - 36\sqrt{6})r^2
\]
which is equivalent to
\[
8R + 3r \geq 2(3\sqrt{6} - 4)R + (35 - 12\sqrt{6})r
\]
or
\[
(16 - 6\sqrt{6})R \geq (32 - 12\sqrt{6})r
\]
which is nothing else than Euler’s inequality, \(R \geq 2r\). □

Concerning this matter, we formulate the following

**Conjecture.** Find the optimal constant \(k\) such that in any acute-angled triangle \(ABC\) with sides of lengths \(a,b,c\) and area \(S\) the following inequality is valid:
\[
a^2 + b^2 + c^2 \leq 4S\sqrt{3} + k[(a-b)^2 + (b-c)^2 + (c-a)^2].
\]

In our research we have found that the constant \(k = \frac{2 - \sqrt{3}}{3 - 2\sqrt{2}}\) is optimal and it is attained for a right angled isosceles triangle, but we do not know if the reversed Finsler-Hadwiger inequality holds true in this case.

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**References**


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