IMPROVED FINSLER-HADWIGER INEQUALITY REVISITED

CEZAR LUPU, VIRGIL NICULA

ABSTRACT. In this note we give another generalized and sharpened version of the Finsler-Hadwiger inequality different from the one given by Wu and Debnath in [6]. Our approach is rather elementary and it doesn’t use auxiliary trigonometric inequalities.

1. INTRODUCTION & MAIN RESULT

Throughout this note we use the notation $\sum$ to express the cyclic sum, for example: $\sum f(A) = f(A) + f(B) + f(C)$.

In [6] it is proved the following generalized sharpened version of the Finsler-Hadwiger inequality, namely

Theorem 1.1. Let $a, b, c$ be the lengths of sides of triangle $ABC$ and let $F, R, d$ denote respectively its area, circumradius and the distance between circumcenter and the incenter. Then for real numbers $\lambda \geq 2$, the following inequality is true

$$\sum a^\lambda \geq 2^{\lambda - \frac{3}{2}} \left(3 + \frac{d^2}{R^2}\right)^{\frac{\lambda}{2}} F^\frac{\lambda}{2} + \sum |a - b|^\lambda.$$

This theorem is an improvement of the following generalized Finsler-Hadwiger inequality given by Wu in [4],

Theorem 1.2. For any triangle $ABC$ we have the inequality

$$\sum a^\lambda \geq 2^{\lambda - \frac{3}{2}} F^\frac{\lambda}{2} + \sum |a - b|^2.$$

For $\lambda = 2$, we obtain the celebrated Finsler-Hadwiger inequality (see [1]), namely

Theorem 1.3. For any triangle $ABC$ we have the inequality

$$\sum a^2 \geq 4\sqrt{3}F + \sum (a - b)^2.$$
Theorem 1.4. Let \( a, b, c \) be the lengths of sides of triangle \( ABC \) and let \( F, R, r, s \) denote respectively its area, circumradius, inradius and the semiperimeter. Then for real numbers \( \lambda \geq 2 \), the following inequality is true
\[
\sum a^\lambda \geq 2^{\lambda - \frac{3}{2}} (KP)^{\frac{3}{2}} F^{\frac{3}{2}} + \sum |a - b|^\lambda,
\]
where \( K = \frac{1}{s} \sqrt{4R^2 + 4Rr + 3r^2} \) and \( P = \sqrt{3 + \frac{4(R - 2r)}{4R + r}} \).

2. Proof of Main result

In the proof of Theorem 1.5, we shall use the following from [2], that is
Lemma 2.1. Let \( x_1, x_2, \ldots, x_n \) be nonnegative numbers, and \( p \geq 1 \). Then
\[
\sum_{i=1}^{n} x_i^p \geq n^{1-p} \left( \sum_{i=1}^{n} x_i \right)^p,
\]
with equality holding if and only if \( x_1 = x_2 = \ldots = x_n \) or \( p = 1 \). Moreover,
\[
(x_1 + x_2)^p \geq x_1^p + x_2^p,
\]
with equality holding if and only if \( x_1 = 0 \) or \( x_2 = 0 \) or \( p = 1 \).

This lemma as a consequence of Holder’s inequality and it is used also to prove other inequalities in [4] and [6]. Now, we give the

Proof of Theorem 1.4 By Lemma 2.1 we have
\[
\sum a^\lambda = \sum ((b-c)^2 + (c+a-b)(a+b-c))^\frac{\lambda}{2} \geq \sum |b-c|^\lambda + (c+a-b)^\frac{\lambda}{2} (a+b-c)^\frac{\lambda}{2}.
\]
Further again, by Lemma 2.1, we have
\[
\sum (c+a-b)^\frac{\lambda}{2} (a+b-c)^\frac{\lambda}{2} \geq 3^{1-\frac{\lambda}{2}} \left( \sum (b+c-a)(c+a-b) \right)^{\frac{\lambda}{2}}.
\]
Thus, we obtain
\[
\sum a^\lambda \geq \sum |b-c|^\lambda + 3^{1-\frac{\lambda}{2}} \left( \sum (b+c-a)(c+a-b) \right)^{\frac{\lambda}{2}}.
\]
Now, we prove that
\[
\sum (b+c-a)(c+a-b) \geq 4F \cdot K \cdot P,
\]
where \( K \) and \( P \) were defined in the previous section. By the well-known equalities in a triangle, \( ab + bc + ca = s^2 + 4Rr + r^2 \) and \( a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \), we deduce that \( \sum (b+c-a)(c+a-b) = 4r(4R + r) \) and we have to prove that
\[
4r(4R + r) \geq 4F \cdot K \cdot P
\]
which is successively equivalent to
\[
4R + r \geq \sqrt{4R^2 + 4Rr + 3r^2} \cdot \sqrt{3 + \frac{4(R - 2r)}{4R + r}},
\]
\[
(4R^2 + 4Rr + 3r^2)(16R - 5r) \leq (4R + r)^3,
\]
and finally equivalent to
\[ 4r(R - 2r)^2 \geq 0. \]

Finally, we obtain
\[ \sum a^\lambda \geq 2^{\lambda - \frac{3}{2}} (KP)^{\frac{1}{2}} F^{\frac{\lambda}{2}} + \sum |a - b|^{\frac{\lambda}{2}}, \]
where \( K = \frac{1}{s} \sqrt{4R^2 + 4Rr + 3r^2} \) and \( P = \sqrt{3 + \frac{4(R - 2r)}{4R + r}}. \qed \)

By the well-known Gerretsen’s inequality, \( s^2 \leq 4R^2 + 4Rr + 3r^2 \) (see [7]), we obtain a generalization of Theorem 5 from [8], namely

Theorem 2.2. For any triangle \( ABC \), we have the inequality
\[ \sum a^\lambda \geq 2^{\lambda - \frac{3}{2}} (KP)^{\frac{1}{2}} F^{\frac{\lambda}{2}} + \sum |a - b|^{\frac{\lambda}{2}}. \]

For \( \lambda = 2 \), Theorems 1.4 and Theorem 2.2 leads to the following refinements of the Finsler-Hadwiger inequality

Corollary 2.3. For any triangle \( ABC \), we have the inequality
\[ \sum a^2 \geq 4F \cdot \frac{1}{s} \sqrt{4R^2 + 4Rr + 3r^2} \cdot \sqrt{3 + \frac{4(R - 2r)}{4R + r}} + \sum (a - b)^2. \]

Corollary 2.4. For any triangle \( ABC \), we have the inequality
\[ \sum a^2 \geq 4F \cdot \sqrt{3 + \frac{4(R - 2r)}{4R + r}} + \sum (a - b)^2. \]

Corollary 2.4 appears also in [8]. The proof was based on a variant of a particular case of Schur’s algebraic inequality.

REFERENCES
Bibliography

University of Bucharest, Faculty of Mathematics, Str. Academiei 14, RO–70109, Bucharest and University of Craiova, Faculty of Mathematics, Str. A.I. Cuza 10, RO–200585, Craiova, Romania
E–mail address: lupucezar@yahoo.com, lupucezar@gmail.com

'Ion Creangă' High School, Str. Str. Cuza Vodă, No. 51, Bucharest, Romania
E–mail address: levinicula@yahoo.com