PROBLEMS AND SOLUTIONS

Edited by Gerald A. Edgar, Doug Hensley, Douglas B. West

Proposed problems and solutions should be sent in duplicate to the MONTHLY problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before September 30, 2012. Additional information, such as generalizations and references, is welcome. The problem number and the solver’s name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11642. Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania. Let \( \alpha, \beta, \gamma \) be positive real numbers, with \( \gamma > 1 \).

(a) Prove that

\[
\lim_{x \to 1^-} (1 - x)^\beta \sum_{n=1}^{\infty} \gamma^{n \alpha} x^n \gamma^n = \begin{cases} 
0 & \text{when } \beta > \alpha, \\
\infty & \text{when } \beta < \alpha.
\end{cases}
\]

(b) Does the limit exist if \( \beta = \alpha \)?

11643. Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA. Let \( r \) be a real number with \( 0 < r < 1 \), and define a discrete probability measure \( P \) on \( \mathbb{N} \) by

\[
P(k) = (1 - r) r^{k-1} \text{ for } k \geq 1.
\]

Show that there are uncountably many triples \((A_1, A_2, A_3)\) of subsets of \( \mathbb{N} \) that are mutually independent, that is,

\[
P(A_i \cap A_j) = P(A_i) P(A_j) \quad \text{for } i \neq j \quad \text{and} \quad P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3).
\]

11644. Proposed by Albert Stadler, Herrliberg, Switzerland. Let \( n \) be a nonnegative integer, and let \( B_j \) be the \( j \)th Bernoulli number, defined for \( j \geq 0 \) by

\[
x/(e^x - 1) = \sum_{k=0}^{\infty} B_k x^k / k!.
\]

Prove that

\[
I_n = \int_0^{\infty} \left( \frac{1}{x^n(e^x - 1)} - \left( \frac{1}{x^n} + \sum_{k=0}^{n} B_k \frac{x^{k-n-1}}{k!} \right) e^{-x} \right) dx.
\]

Prove that \( I_0 = \gamma - 1 \), that \( I_1 = 1 - (1/2) \log(2\pi) \), and that for \( n \geq 1 \),

\[
I_{2n} = (\log(2\pi) + \gamma) \frac{B_{2n}}{(2n)!} + (-1)^n \frac{2\zeta'(2n)}{(2\pi)^{2n}} + \frac{1}{2(2n-1)!} H_{2n-1} - \sum_{k=0}^{n-1} B_{2k} \frac{H_{2n-2k}}{(2k)!}.
\]

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and that for \( n \geq 1, \)
\[
I_{2n+1} = (-1)^n \zeta(2n + 1) \frac{1}{2(2n)!} H_{2n} + \sum_{k=0}^{n} \frac{B_{2k}}{(2k)!} \cdot \frac{H_{2n+1-2k}}{(2n + 1 - 2k)!}.
\]
Here, \( H_n \) denotes \( \sum_{k=1}^{n} 1/k, \) \( \zeta, \) the Riemann zeta function, and \( \gamma, \) Euler’s constant.

11645. Proposed by Christopher J. Hillar, University of California, Berkeley, CA, Lionel Levine, Cornell University, Ithaca, NY, and Darren Rhea, University of California San Francisco, San Francisco, CA. Determine all positive integers \( n \) such that the polynomial \( g \) in two variables given by \( g(x, y) = 1 + y^2 \sum_{k=1}^{n} x^{2k} + y^4 x^{2n+2} \) factors in \( \mathbb{C}[x, y]. \)

11646. Proposed by Pál Péter Dálayay, Szeged, Hungary. Let \( ABC \) be an acute triangle, and let \( A_1, B_1, C_1 \) be the intersection points of the angle bisectors from \( A, B, C \) to the respective opposite sides. Let \( R \) and \( r \) be the circumradius and the inradius of \( ABC, \) and let \( R_A, R_B, R_C \) be the circumradii of the triangles \( AC_1B_1, BA_1C_1, \) and \( CA_1B_1, \) respectively. Let \( H \) be the orthocenter of \( ABC, \) and let \( d_a, d_b, d_c \) be the distances from \( H \) to sides \( BC, CA, \) and \( AB, \) respectively. Show that \( 2r(R_A + R_B + R_C) \geq R(d_a + d_b + d_c). \)

11647. Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA and Tudorel Lupu, Decebal High School, Constanta, Romania. For continuous \( \Psi \) on \([0, 1], \) let \( V \Psi \) be the function on \([0, 1] \) given by \( V \Psi(t) = \int_{0}^{t} \Psi(x) \, dx. \) For \( \phi \) a differentiable function from \([0, 1] \) to \( \mathbb{R}, \) satisfying \( \phi'(x) \neq 0 \) for \( 0 < x < 1, \) let \( V \phi \Psi(t) = \int_{0}^{t} \phi(x) \Psi(x) \, dx. \) Show that if \( f \) and \( g \) are continuous real-valued functions on \([0, 1] \) then there exists \( x_0 \in (0, 1) \) such that
\[
(V \phi f)(x_0) \int_{0}^{1} g(x) \, dx - (V \phi g)(x_0) \int_{0}^{1} f(x) \, dx
= \phi(0) \left(V f(x_0) \int_{0}^{1} g(x) \, dx - V g(x_0) \int_{0}^{1} f(x) \, dx \right).
\]

11648. Proposed by Moubinool Omarjee, Paris, France. Let \( E \) be the set of all continuous, differentiable functions from \((0, 1]\) into \( \mathbb{R} \) such that \( \int_{0}^{1} t^{1/2} f^2(t) \, dt \) converges. Let \( F \) be the set of all \( f \) in \( E \) such that \( \int_{0}^{1} t^{-3/2} f^2(t) \, dt \) and \( \int_{0}^{1} t^{1/2} f^2(t) \, dt \) converge. Equip \( E \) with the distance
\[
d(f, g) = \left( \int_{0}^{1} t^{1/2} (f - g)^2(t) \, dt \right)^{1/2}
\]
to make it a metric space. Is \( F \) a closed subset of \( E? \)

**SOLUTIONS**

**Piercing Many Segments**

11507 [2010, 459]. Proposed by Marius Cavachi, “Ovidius” University of Constanta, Constanta, Romania. Let \( n \) be a positive integer and let \( R \) be a plane region of perimeter 1. Inside \( R \) there are a finite number of line segments, the sum of whose lengths
is greater than \( n \). Prove that there exists a line that intersects at least \( 2n + 1 \) of the segments.

**Solution by Jim Simons, Cheltenham, U. K.** We may assume that \( R \) is convex, for otherwise we can take its convex hull, which will have perimeter less than 1, and then dilate about a point in its interior to create a convex region with perimeter 1 that still includes all the line segments. For a bounded convex set \( C \), we define \( W_\theta(C) \), the width of \( C \) in direction \( \theta \), to be the minimum width of a strip in the plane that includes \( C \) and that is bounded by two parallel lines making angle \( \theta \) with the \( x \)-axis.

Represent a point \( z \) on the boundary of \( R \) as the pair \( (t, \gamma) \), where \( t \) is the arc length along the boundary from a fixed starting point and \( \gamma \) is the angle between the \( x \)-axis and the tangent to the boundary of \( R \) at \( z \). It follows that

\[
W_\theta(R) = \frac{1}{2} \oint |\sin(\theta - \gamma)| \, dt.
\]

We compute the average value of \( W_\theta(R) \) over all angles \( \theta \):

\[
\frac{1}{\pi} \int_0^\pi W_\theta(R) \, d\theta = \frac{1}{2\pi} \int_0^\pi \oint |\sin(\theta - \gamma)| \, dt \, d\theta = \frac{1}{2\pi} \oint \int_0^\pi |\sin(\theta - \gamma)| \, d\theta \, dt = \frac{1}{\pi} \oint dt = \frac{1}{\pi}.
\]

Similarly, if \( S \) is the given set of line segments, and if \( s \in S \) has length \( |s| \) and makes an angle \( \gamma(s) \) with the \( x \)-axis, then

\[
\sum_{s \in S} W_\theta(s) = \sum_{s \in S} |s| |\sin(\theta - \gamma(s))|.
\]

Averaging this quantity over all angles, we obtain

\[
\frac{1}{\pi} \int_0^\pi \sum_{s \in S} W_\theta(s) \, d\theta = \frac{1}{\pi} \int_0^\pi \sum_{s \in S} |s| |\sin(\theta - \gamma(s))| \, d\theta = \frac{1}{\pi} \sum_{s \in S} |s| \int_0^\pi |\sin(\theta - \gamma(s))| \, d\theta = \frac{2\sigma}{\pi},
\]

where \( \sigma = \sum_{s \in S} |s| > n \). It follows that there is some angle \( \theta \) for which \( \sum_{s \in S} W_\theta(s) \geq 2\sigma W_\theta(R) \). Now consider the lines that make angle \( \theta \) with the \( x \)-axis and that intersect \( R \). They lie in the strip we have defined, and we can index them by the distance \( u \) from one edge of the strip. Let \( n(u) \) be the number of segments from \( S \) that intersect such a line. Now

\[
\sum_{s \in S} W_\theta(s) = \int_0^{W_\theta(R)} n(u) \, du.
\]

We have \( n(u) \geq 2\sigma \) for some \( u \), and hence \( n(u) \geq 2n + 1 \).

Also solved by R. Chapman (U. K.), O. P. Lossers (Netherlands), T. Starbird, R. Stong, GCHQ Problems Group (U. K.), and the proposer.

**Perfect Squares with Specified Differences**

**11508** [2010, 459]. Proposed by Mih’aly Bencze, Brasov, Romania. Prove that for all positive integers \( k \) there are infinitely many positive integers \( n \) such that \( kn + 1 \) and \( (k + 1)n + 1 \) are both perfect squares.

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Solution by Yury J. Ionin, Champaign, IL. We prove a more general result: If \(a\) and \(b\) are positive integers such that \(ab\) is not a perfect square, then for all integers \(c\) and \(d\) there are infinitely many positive integers \(n\) such that \(an + c^2\) and \(bn + d^2\) are both perfect squares.

Letting \(an + c^2 = x^2\) and \(bn + d^2 = y^2\), we have


d\(x^2 - ay^2 = bc^2 - ad^2\). \hspace{1cm} (1)

Conversely, if \((x, y)\) is a solution to (1) and \(x^2 \equiv c^2 \pmod{a}\), then, for \(n = (x^2 - c^2)/a\), we have \(an + c^2 = x^2\) and \(bn + d^2 = y^2\).

Let \((u, v)\) be a solution to the Pell equation


d\(u^2 - abv^2 = 1\), \hspace{1cm} (2)

and let \(x = cu + adv\) and \(y = du + bcv\). Now \((x, y)\) is a solution to (1) and \(x^2 \equiv c^2 u^2 \equiv c^2 \pmod{a}\). To complete the proof, note that since \(ab\) is positive and is not a perfect square, the equation (2) has infinitely many solutions.

Also solved by G. Apostolopoulos (Greece), B. D. Beasley, D. Beckwith, R. Chapman (U. K.), and H. M. Choe & E. Jee & S. Kim (S. Korea).

A Combinatorial Identity

11509 [2010, 558]. Proposed by William Stanford, University of Illinois-Chicago, Chicago, IL. Let \(m\) be a positive integer. Prove that

\[
\sum_{k=m}^{m^2-m+1} \frac{(\binom{m^2-2m+1}{k-m})}{k!} = \frac{1}{m!} \cdot \frac{1}{C_m}\left(\frac{m^2}{m}\right) m^{2m-1}.
\]

Solution I by Kim McInturff, Santa Barbara, CA. Among the \(\binom{m^2}{2m-1}\) subsets of size \(2m - 1\) in \(\{1, \ldots, m^2\}\), exactly \(\binom{k-1}{m-1}\binom{m^2-k}{m-1}\) have \(k\) as the median element. Therefore

\[
\sum_{k=m}^{m^2-m+1} \frac{(\binom{m^2-2m+1}{k-m})}{k!} = \frac{(m^2 - 2m + 1)!}{(m^2)!} \sum_{k=m}^{m^2-m+1} \frac{(k-1)!}{(k-m)!} \cdot \frac{(m^2-k)!}{(m^2-m+1-k)!}
\]

\[
= \frac{(m^2 - 2m + 1)!}{(m^2)!} \cdot \frac{(m-1)!}{2} \sum_{k=m}^{m^2-m+1} \frac{(k-1)!}{(k-m)!} \cdot \frac{(m^2-k)!}{m!}.
\]

Solution II by Takis Konstantopoulos, Uppsala University, Uppsala, Sweden. Let \(A, B, a,\) and \(b\) be integers such that \(A \geq 1, B \geq 1, a \geq b,\) and \(B - A \geq a - b\). We prove the more general identity

\[
\sum_{k=a}^{A-a} \frac{\binom{A}{k-a}}{\binom{B}{k-b}} = \frac{B+1}{(a-b+1)\binom{B-A+1}{a-b+1}}. \hspace{1cm} (*)
\]

The desired result follows by setting \(A = (m-1)^2, B = m^2 - 1, a = m,\) and \(b = 1\) and dividing by \(m^2\). To prove (*), recall the beta integral

\[
\int_0^1 t^p(1-t)^q dt = \frac{p!q!}{(p+q+1)!}.
\]

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for nonnegative integers \( p \) and \( q \). Using this, we write
\[
\frac{1}{\binom{R}{k}} = (B + 1) \int_0^1 t^{k-b}(1-t)^{B+b-k} \, dt
\]
when \( b \leq k \leq B + b \) and in particular when \( a \leq k \leq A + a \). The sum in (\( * \)) then becomes
\[
\sum_{k=a}^{A+a} \binom{A}{k-a} \binom{B}{k-b} = (B + 1) \int_0^1 t^{-b}(1-t)^{B+b} \sum_k \binom{A}{k-a} \binom{1}{k}^k \, dt
\]
\[
= (B + 1) \int_0^1 t^{-b}(1-t)^{B+b} \left( \frac{t}{1-t} \right)^a \left( 1 + \frac{t}{1-t} \right)^A \, dt
\]
\[
= (B + 1) \int_0^1 t^{a-b}(1-t)^{B-A+b-a} \, dt = \frac{B + 1}{(a-b + 1)\binom{B-A+1}{a-b+1}}.
\]

**Solution III by Stanley Xiao, University of Waterloo, Waterloo, Ontario, Canada.** We generalize the problem: given positive integers \( R, B, \) and \( b \) with \( b \leq B \), we show
\[
\sum_{r=0}^R \frac{\binom{R}{b} \binom{R}{b+r}}{(b + r) \binom{b+r}{R}} = 1. \tag{**}
\]
The desired result follows by setting \( B = 2m - 1 \), \( R = (m - 1)^2 \), and \( b = m \).

To prove (**) consider a deck of \( B \) blue cards and \( R \) red cards. A game is played where the player pulls cards without replacement from a shuffled deck and wins as soon as he obtains \( b \) blue cards. The probability of winning is 1, since \( b \leq B \). We compute the probability that the player wins after drawing exactly \( r \) red cards, with \( 0 \leq r \leq R \). The probability that exactly \( b \) of the first \( b + r \) cards are blue is \( \binom{R}{b} \binom{R}{b+r} / \binom{b+r}{b+r} \). The probability that the last card is blue given that exactly \( b \) of the first \( b + r \) cards are blue is \( b/(b + r) \). Hence the probability of the player winning after drawing exactly \( r \) red cards is
\[
\frac{\binom{R}{b} \binom{R}{b+r}}{(b + r) \binom{b+r}{R}}.
\]
Summing over \( r \) gives (**).

Also solved by M. Anton & E. Niehaus & E. Shirley, M. Bataille (France), D. Beckwith, R. Chapman (U. K.), R. Cheplyaka & V. Lucic & L. Pebody, P. De (India), M. N. Deshpande (India), M. Goldenberg & M. Kaplan, O. Kouba (Syria) M. E. Larsen (Denmark), J.-Y. Lee (Korea), O. P. Lossers (Netherlands), Á. Plaza (Spain), O. G. Ruehr, J. Schlosberg, J. Simons (U. K.), N. C. Singer, S. Song (Korea), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), M. Vowe (Switzerland), F. Vrabec (Austria), H. Widmer (Switzerland), BSI Problems Group (Germany), Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), Mathramz Problem Solving Group, and the proposer.

**The Inf of the Circumcenter-Centroid-Incenter Angle is \( \pi/2 \)**

**11516** [2010, 649]. **Proposed by Elton Bojaxhiu, Albania, and Enkel Hysnelaj, Australia.** Let \( \mathcal{T} \) be the set of all nonequilateral triangles. For \( T \) in \( \mathcal{T} \), let \( O \) be the circumcenter, \( Q \) the incenter, and \( G \) the centroid. Show that \( \inf_{T \in \mathcal{T}} \angle O G Q = \pi/2 \).

**Editorial comment.** As pointed out by O. Geupel (Germany) and B. Mulansky (Germany), the solution to this problem was actually contained in: Andrew P. Guinand,
“Euler Lines, Tri-tangent Centers, and Their Triangles” in *Amer. Math. Monthly* 91 (1984) 290–300. In that article, Guinand defines the “critical circle” of a triangle as that for which the segment between the centroid and the orthocenter is a diameter. (This is also known as the “Circle of Bellot-Rosada”.)

Guinand’s Theorem 1 (p. 291) states: “The incenter of a non-equilateral triangle lies inside the critical circle, and all the excenters lie outside it.” Thus we have immediately that the inf desired in this problem cannot be less than \( \pi/2 \).

Guinand’s Theorem 4 (p. 296) states, in part, that: “Every point inside the critical circle except the nine-point center is the incenter of some triangle.” (He assumes \( G \) and \( O \) are fixed, hence so is the critical circle.) The nine-point center must be exempted because, in some sense, it corresponds to the equilateral triangle. This clause of Theorem 4 guarantees that \( \angle O G Q \) can be made arbitrarily close to \( \pi/2 \) by making \( Q \) close to \( G \) and even closer to the boundary of the critical circle.

Two readers, J.-P. Grivaux (France) and J. Schlosberg, pointed out that the claim that \( \angle O G Q \) is always obtuse follows from Problem 10955, *Amer. Math. Monthly* 111 (2004) 67–69. Solved by R. Bagby, C. Curtis, P. P. Dályay (Hungary), O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, L. R. King, O. Kouba (Syria), K. Mcinturff, B. Mulansky (Germany), C. R. Pranesachar (India), J. Schlosberg, R. Stong, M. Tetoiva (Romania), Z. Vörös (Hungary), J. B. Zacharias, Barclays Capital Quantitative Analytics Group (U. K.), and the proposers.

A Harmonious Sum

11519 [2010, 649]. *Proposed by Ovidiu Furdui, Câmpia Turzii, Cluj, Romania*. Find

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} \frac{H_{n+m}}{n + m},
\]

where \( H_n \) denotes the \( n \)th harmonic number.

**Solution I by Wim Nuij, Eindhoven, The Netherlands.** We show that the value is

\[
\frac{\pi^2}{12} - \frac{\log 2}{2} - \frac{\log^2 2}{2}.
\]

Since the sum is not absolutely convergent, instead we consider

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} \frac{H_{n+m}}{(n + m)}.
\]

This series is absolutely convergent for \( |x| < 1 \), so combining the terms where \( n + m = k + 1 \) yields

\[
\sum_{k=0}^{\infty} (-1)^{k+1} x^{k+1} \frac{k}{k+1},
\]

which can be split into

\[
\sum_{k=0}^{\infty} (-1)^{k+1} x^{k+1} H_{k+1} + \sum_{k=1}^{\infty} (-1)^{k} x^{k+1} \frac{H_k}{k+1} + \sum_{k=0}^{\infty} (-1)^{k} x^{k+1} \frac{1}{(k+1)^2}.
\]

The power series of \( \log(1 + x) \) is \( \sum_{k=0}^{\infty} (-1)^k x^{k+1}/(k + 1) \), so

\[
\frac{\log(1 + x)}{1 + x} = \sum_{k=0}^{\infty} (-1)^k x^{k+1} \sum_{i=0}^{k} \frac{1}{i + 1} = \sum_{k=0}^{\infty} (-1)^k x^{k+1} H_{k+1}.
\]

Integration leads to

\[
\frac{\log^2(1 + x)}{2} = \sum_{k=1}^{\infty} (-1)^{k-1} H_k \frac{x^{k+1}}{k+1},
\]
so (1) equals
\[ -\frac{\log(1+x)}{1+x} - \frac{\log^2(1+x)}{2} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{(k+1)^2}. \]

For \( x \to 1^- \) this tends to \(-\frac{\log 2}{2} - \frac{\log^2 2}{2} + \frac{\pi^2}{12}\), but we need to justify that this limit is the sum of the original series.

Let \( T_n(x) = \sum_{m=1}^{\infty} (-1)^{m-1} x^{n+m} H_{n+m}/(n+m) \). Since \( H_p/p \) is strictly decreasing and \( \lim_{p \to \infty} H_p/p = 0 \), the alternating series \( T_n(x) \) converges uniformly on \([0, 1]\), so it is continuous on this interval. For \( 0 < x \leq 1 \) we have
\[ \frac{H_{p-1}}{p-1} - x \frac{H_p}{p} > x \frac{H_p}{p} - x^2 \frac{H_{p+1}}{p+1} > 0 \quad \text{for all } p > 1, \]
where the first inequality follows from the discriminant
\[ \left( \frac{H_p}{p} \right)^2 - \frac{H_p - \frac{1}{p}}{p-1} \cdot \frac{H_p + \frac{1}{p}}{p+1} < 0, \]
and the second inequality follows from \( H_p/p \) being strictly decreasing. Hence \( T_n(x) > T_{n+1}(x) > 0 \) for \( 0 < x \leq 1 \), implying that \( T_n(x) \to 0 \) uniformly. Thus the alternating series \( \sum_{n=1}^{\infty} (-1)^{n-1} T_n(x) \) converges uniformly on \([0, 1]\). Its sum is continuous at \( x = 1 \), justifying the taking the limit.

Solution II by Richard Stong, San Diego, CA. Let \( S \) denote the desired sum. Since the inner series is alternating with terms decreasing in magnitude, we have
\[ \left| \sum_{m=1}^{\infty} (-1)^{m+n} \frac{H_{n+m}}{n+m} \right| \leq \frac{H_{n+1}}{n+1} \to 0 \]
as \( n \to \infty \). Thus the terms in the outer sum tend to 0. Hence it suffices to show that even-indexed partial sums of \( S \) converge, and we may add pairs of consecutive terms (say the \((2r-1)\)-st and \((2r)\)-th). Doing the same for the inner sum as well gives
\[ S = \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{H_{2r+m-1}}{2r+m-1} + (-1)^m \frac{H_{2r+m}}{2r+m} \]
\[ = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{H_{2r+2s-2}}{2r+2s-2} - \frac{2H_{2r+2s-1}}{2r+2s-1} + \frac{H_{2r+2s}}{2r+2s} \]
\[ = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{2H_{2r+2s} - 3}{(2r+2s-2)(2r+2s-1)(2r+2s)}. \]

This sum now converges absolutely. Rearrange it by letting \( t = r + s \) and noting that each value of \( t \) occurs for \( t-1 \) pairs \((r, s)\) (and include the vanishing term where \( t = 1 \)) to get
\[ S = \sum_{t=1}^{\infty} (2H_{2t} - 3)/(4t(2t-1)). \]
Applying the same regularization procedure to the well-known identities
\[ \sum_{t=1}^{\infty} (-1)^{t-1} \frac{H_t}{n} = \int_0^1 \frac{1 - \log(1-x)}{1+x} \, dx = \frac{\pi^2}{12} - \frac{\log^2 2}{2}, \]

\[ \sum_{t=1}^{\infty} (-1)^{t-1} \frac{1}{n} = \log 2 \]

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gives
\[
\sum_{i=1}^{\infty} \frac{H_{2i} - 1}{2t(2t-1)} = \frac{\pi^2}{12} - \frac{\log^2 2}{2}, \quad \sum_{i=1}^{\infty} \frac{1}{2t(2t-1)} = \log 2.
\]

Comparing these formulas yields \( S = \frac{\pi^2}{12} - \frac{\log^2 2}{2} \).

**Editorial comment.** A number of incomplete solutions were received. Most found
\[ \lim_{k \to \infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-x)^{n+m} H_{n+m}/(n + m) \]
(as in Solution I) and tried to invoke Abel’s theorem to argue that it is the sum of the series. However, this requires
the rearrangement of the terms of the double series into a single series, so the
use of the limit needs to be justified. For example, the series \( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} \)
with \( a_{n,1} = 1, a_{n,2} = -1 \), and \( a_{n,m} = 0 \) for \( n \geq 1 \) and \( m \geq 3 \) has sum 0, but
\[ \lim_{k \to \infty} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} x^{n+m} = 1. \]

Mark Wildon showed that if \( \langle a_n \rangle \) is a decreasing sequence of positive numbers
with limit 0 as \( n \to \infty \) such that \( (a_n - a_{n+1}) \) and \( a_n/a_{n+1} \) are also decreasing, then
\[ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} a_{n+m} = \lim_{k \to \infty} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-x)^{n+m} a_{n+m}. \]
Summing the original series over the diagonal \( k = m + n \) yields a divergent series.


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**A Short Proof that a Factor Ring of a PID Is Armendariz**

A commutative ring \( R \) is said to be Armendariz if, \( f = \sum_{i=0}^{m} a_i X^i, \ g = \sum_{j=0}^{n} b_j X^j \in R[X] \) with \( fg = 0 \) implies \( a_i b_j = 0 \) for all \( i, j \).

The Gauss lemma shows \( A/I \) is Armendariz for a PID \( A \) and an ideal \( I \).

Indeed, recall over a PID \( A \), the ideal \( c(f) \) generated by the coefficients of a polynomial \( f \in A[X] \) is called the content ideal and, Gauss’s lemma says that \( c(fg) = c(f)c(g) \). Now, if \( f = \sum_{i=0}^{m} a_i X^i, \ g = \sum_{j=0}^{n} b_j X^j \in A[X] \) with \( fg \in I[X] \), then writing \( l, m \) for the GCD of the \( a_i \)’s and the \( b_j \)’s, respectively, we have the content ideals \( c(f) = (l), c(g) = (m) \). So, for each \( i, j \),
\[ a_i b_j \in (l)(m) = c(f)c(g) = c(fg) \subseteq I. \]

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