\[ \sin x \text{ if } x < \pi/2 \text{ and } f(x) = 1 \text{ if } x \geq \pi/2. \] Show that \( \lim_{n \to \infty} x_n \sqrt{n} = \sqrt{3}. \)

(e) Finally, suppose instead that \( 0 < x_1 < 1 \) and \( f(x) = 1 - e^{-x}. \) Show that, in this case, \( \lim_{n \to \infty} n x_n = 2. \)

11245. Proposed by Cezar Lupu, University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanza, Romania. Consider an acute triangle with sides of lengths \( a, b, \) and \( c, \) and with an inradius of \( r \) and a circumradius of \( R. \) Show that

\[
\frac{r}{R} \leq \frac{\sqrt{2}(2a^2 - (b - c)^2)(2b^2 - (c - a)^2)(2c^2 - (a - b)^2)}{(a + b)(b + c)(c + a)}.
\]

11246. Proposed by Lee Goldstein, Wichita, KS. Let \( J_n \) be the \( n \)th Bessel function of the first kind, and let \( \text{K}_n \) be the \( n \)th modified Bessel function of the second kind (also known as a Macdonald function), defined by

\[
\text{K}_n(z) = \frac{\Gamma([n] + 1/2)(2z)^{|n|}}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{(t^2 + z^2)^{|n|+1/2}} dt.
\]

Show that, for any positive \( b \) and any real \( \lambda, \)

\[
\frac{\sqrt{\pi}}{2\sqrt{b}} e^{\lambda^2 - 2b} = \sum_{-\infty}^\infty J_{2n}(4\lambda \sqrt{b}) \text{K}_{n+1/2}(2b).
\]

11247. Proposed by Jürgen Eckhoff, University College London, London, U. K. Let \( A, \) \( B, C, \) and \( D \) be distinct points in the plane with the property that any three of them can be covered by some strip of width 1. Show that there is a strip of width \( \sqrt{2} \) covering all four points, and demonstrate that if no strip of width less than \( \sqrt{2} \) covers all four, then the points are the corners of a square of side \( \sqrt{2}. \) (A strip of width \( w \) is the closed set of points bounded by two parallel lines separated by distance \( w. \))

11248. Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary. Let \( n \) be a positive integer, and let \( f \) be a continuous real-valued function on \([0, 1]\) with the property that \( \int_0^1 x^k f(x) \, dx = 1 \) for \( 0 \leq k \leq n - 1. \) Prove that \( \int_0^1 (f(x))^2 \, dx \geq n^2. \) (The case \( n = 2 \) appeared in the 55th Romanian Mathematical Olympiad, Gazeta Matematică 5 (2004), 219.)

11249. Proposed by David Beckwith, Sag Harbor, NY. A node-labeled rooted tree is a tree such that any parent with label \( \ell \) has \( \ell + 1 \) children, labeled \( 1, 2, \ldots, \ell + 1, \) and such that the root vertex (generation 0) has label 1. Find the population of generation \( n. \)

SOLUTIONS

How Many Powers of 2 Must Survive?

11114 [2004, 823]. Proposed by Benne de Weger, Technical University of Eindhoven, Eindhoven, Netherlands. For nonzero \( x \in \mathbb{Q}, \) let \( \text{ord}_2(x) \) be the exponent of 2 in the prime factorization of \( x. \) For \( n \in \mathbb{N}, \) let \( A_n = \sum_{k=1}^n \frac{2^k}{k}. \)