Problem 1. Prove that there is no smallest positive number. Does there exist a positive rational number? Given a real number \(x\), does there exist a smallest real number \(y > x\)?

Problem 2. Given \(y \in \mathbb{R}, n \in \mathbb{N}, \) and \(\epsilon > 0,\) show that for some \(\delta > 0,\) if \(u \in \mathbb{R}\) and \(|u - y| < \delta,\) then \(|u^n - y^n| < \epsilon.\)

Problem 3. a) Prove that the ellipsoid
\[
E = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}
\]
is convex.
b) Prove that all boxes in \(\mathbb{R}^m\) are convex.

Problem 4. A function \(f : (a, b) \to \mathbb{R}\) is said to be convex if for all \(x, y \in (a, b)\) and all \(s, t \in [0, 1]\) with \(s + t = 1,\)
\[
f(sx + ty) \leq sf(x) + tf(y).
\]
a) Prove that \(f\) is convex if and only if the set of points \(S\) above its graph is convex in \(\mathbb{R}^2.\)
b) Prove that a convex function is continuous.
c) Suppose that \(f\) is convex and \(a < x < u < b.\) The slope \(\sigma\) of the line through \((x, f(x))\) and \((u, f(u))\) depends on \(x\) and \(u,\) \(\sigma = \sigma(x, u).\) Prove that \(\sigma\) increases when \(x\) increases, and \(\sigma\) increases when \(u\) increases.
d) Suppose \(f\) is second order differentiable. That is, \(f\) is differentiable and \(f'\) is also differentiable. As is standard, write \((f')' = f''\). Prove that \(f\) is convex if and only if \(f'' \geq 0.\)
e) Formulate a definition of convexity for a function \(f : M \to \mathbb{R}\) where \(M \subset \mathbb{R}^m\) is a convex set.

Problem 5. Let \((a_n)_{n \geq 1}\) be a sequence of real numbers. It is bounded if the set \(A = \{a_1, a_2, \ldots\}\) is bounded. The limit supremum, or \(\limsup\) of \((a_n)\) as \(n \to \infty\) is
\[
\limsup_{n \to \infty} a_n = \lim_{n \to \infty} \left( \sup_{k \geq n} a_k \right).
\]
a) If the sequence \((a_n)_n\) is bounded, why does \(\limsup\) exist?
b) If \(\sup\{a_n\} = \infty\), what is \(\limsup_{n \to \infty} a_n\)?
c) If \(\lim_{n \to \infty} a_n = -\infty\), how should we define \(\limsup a_n\)?
d) When is it true that
\[
\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n,
\]
\[
\limsup_{n \to \infty} (ca_n) = c \limsup_{n \to \infty} a_n?
\]
e) Define the limit infimum or \(\liminf\) of a sequence of numbers, and find a formula relating it to limit supremum.

**Problem 6.** Let \(x_1 = a > 0\) and for \(n = 1, 2, \ldots\), let
\[
x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right), \quad n \geq 1.
\]
Prove that the sequence converges and find its limit.

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**Problem 7.** Let \(a_1, a_2, \ldots, a_n\) and \(b_1, b_2, \ldots, b_n\) be positive real numbers. Show that
\[
\prod_{i=1}^{n} (a_i + b_i)^{1/n} \geq \prod_{i=1}^{n} a_i^{1/n} + \prod_{i=1}^{n} b_i^{1/n}.
\]

**Problem 8.** For each \(n \geq 1\), define the sequence \((x_n)_{n \geq 1}\) by
\[
x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n, \quad n \geq 1.
\]
Show that \(x_n\) is convergent as \(n \to \infty\).

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**Problem 9.** Let \(0 < x_1 < 1\) and \(x_{n+1} = x_n(1 - x_n), \quad n = 1, 2, 3, \ldots\). Show that the sequence \((x_n)_{n \geq 1}\) converges to 0.

**Problem 10.** Given \(x > 0\) and \(n \in \mathbb{N}\), prove that there is a unique \(y > 0\) such that \(y^n = x\). That is, \(y = \sqrt[n]{x}\) exists and is unique.