# ALGEBRAIC-GEOMETRIC PROOFS OF THE WEITZENBOCK AND FINSLER-HADWIGER INEQUALITIES REVISITED

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## 1. Introduction & The proofs

Many of the most important results in theory of geometric inequalities were discovered by the beginning of the 20-th century. The first important inequality was discovered in 1919 and it is due to Weitzenbock (see [2]),

**Theorem 1.1.** *In any triangle $ABC$ with sides of lenghts $a, b, c$, the following inequality holds*

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3},$$

*where $S$ denotes the area of the triangle $ABC$.*

The above theorem also appeared in International Mathematical Olympiad in 1961 and many proofs if it can be found in [1]. A refinement of Theorem 1.1 is the Finsler-Hadwiger inequality

**Theorem 1.2.** *In any triangle $ABC$ with sides of lenghts $a, b, c$, the following inequality holds*

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3} + (a-b)^2 + (b-c)^2 + (c-a)^2,$$

*where $S$ denotes the area of the triangle $ABC$.*

Many proofs of Theorem 1.2 can be found in [1] as well as other new proofs in [7], [5] or [6]. In this short note, we shall give more direct proofs for Theorem 1.1 and Theorem 1.2 based on the same idea provided in [5]. As it has been noticed in [5], the Fermat point of a triangle $ABC$ is the point $F$ in or on the triangle for which the sum $AF + BF + CF$ is a minimum. When each of the angles of the triangle is smaller than $120°$, the point $F$ is the point of intersection of the lines connecting the vertices $A, B$, and $C$ to the vertices of equilateral triangles constructed outwardly on the sides of the triangle. In this case the first Fermat point is the interior point $F$ from which each side subtends an angle of $120°$, i.e.,

$$\angle BFC = \angle CFA = \angle AFB = 120°.$$

When one of the vertices of triangle $ABC$ measures $120°$ or more, then a vertex is the Fermat point. For simplity we denote $FA = x, FB = y$ and $FC = z$, where $x, y, z > 0$. If one of angles of the triangle $ABC$ is greater or equal than $120°$, for example $\angle C = 120°$, we have $z = 0, x = b$ and $y = a$ and thus Theorem 1.1 and 1.2

are obvious. In what follows, we consider the case where each angle of the triangle $ABC$ is less than $120°$. Now, we ready to prove the main theorems. We start with

*Proof of Theorem 1.1* By applying the cosine law in the triangles $AFB, BFC$ and $CFA$, we obtain that $a = \sqrt{x^2+xy+y^2}, b = \sqrt{y^2+yz+z^2}, c = \sqrt{z^2+zx+x^2}$ and the area of the triangle $ABC$ is given by

$$S = S_{AFB} + S_{BFC} + S_{CFA} = \frac{\sqrt{3}}{4}(xy+yz+zx).$$

Now, Theorem 1.1 reduces to

$$\sum_{cyc}(x^2+xy+y^2) \geq 3(\sum_{cyc} xy)$$

which is equivalent to

$$(x-y)^2+(y-z)^2+(z-x)^2 \geq 0,$$

with equality if and only if $x=y=z$ which is equivalent to $a=b=c$. $\square$

*Proof of Theorem 1.2* The Finsler-Hadwiger inequality is equivalent to

$$\sum_{cyc}(x^2+xy+y^2) \geq 3(\sum_{cyc} xy)+2\sum_{cyc}(x^2+xy+y^2)-2\sum_{cyc}\sqrt{(x^2+xy+y^2)(y^2+yz+z^2)}$$

which is equivalent to

$$\sum_{cyc}\sqrt{(x^2+xy+y^2)(y^2+yz+z^2)} \geq (x+y+z)^2.$$

On the other hand, it is easy to see that the following equality holds

$$\sum_{cyc}\sqrt{(x^2+xy+y^2)(y^2+yz+z^2)} = \frac{1}{4}\sum_{cyc}\sqrt{(3(z+x)^2+(z-x)^2)\cdot(3(x+y)^2+(x-y)^2)}.$$

By applying Cauchy-Schwarz inequality, we obtain

$$\frac{1}{4}\sum_{cyc}\sqrt{(3(z+x)^2+(z-x)^2)\cdot(3(x+y)^2+(x-y)^2)} \geq \frac{1}{4}\sum_{cyc}(3(z+x)(x+y)+(x-z)(x-y)).$$

Since

$$\frac{1}{4}\sum_{cyc}(3(z+x)(x+y)+(x-z)(x-y)) = (x+y+z)^2,$$

the conclusion follows immediately. $\square$

## References


[1] A. Engel, *Problem Solving Strategies*, Springer Verlag, 1998.

[2] R. Weitzenbock, *Uber eine Ungleichung in der Dreiecksgeometrie*, Math. Z., **5** (1919), 137–146.

[3] P. Finsler, H. Hadwiger, *Einige Relationen im Dreick*, Comm. Mat. Helv., **10** (1937), 316–326.

[4] S. Savchev, T. Andreescu, *Mathematical Miniatures*, Mathematical Association of America, 2002.

[5] C. Alsina, R. Nelsen, *Geometric proofs of the Weitzenbock and Finsler-Hadwiger inequality*, Math. Mag., **81** (2008), 216–219.

[6] D. Marinescu, M. Monea, M. Opincariu, M. Stroe, *A sequence of triangles and geometric inequalities*, Forum Geom., **9** (2009), 291–295.

[7] C. Lupu, *An elementary proof of the Hadwiger-Finsler inequality*, Arhimede, **3** (2003), 18–19.



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