PROBLEMS

11775. Proposed by Isaac Sofair, Fredericksburg, VA. Let $A_1, \ldots, A_k$ be finite sets. For $J \subseteq \{1, \ldots, k\}$, let $N_J = \left| \bigcup_{j \in J} A_j \right|$, and let $S_m = \sum_{J:|J|=m} N_J$.

(a) Express in terms of $S_1, \ldots, S_k$ the number of elements that belong to exactly $m$ of the sets $A_1, \ldots, A_k$.

(b) Same question as in (a), except that we now require the number of elements belonging to at least $m$ of the sets $A_1, \ldots, A_k$.

11776. Proposed by David Beckwith, Sag Harbor, NY. Given urns $U_1, \ldots, U_n$ in a line, and plenty of identical blue and identical red balls, let $a_n$ be the number of ways to put balls into the urns subject to the conditions that

(i) each urn contains at most one ball,

(ii) any urn containing a red ball is next to exactly one urn containing a blue ball, and

(iii) no two urns containing a blue ball are adjacent.

(a) Show that

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1 + t + 2t^2}{1 - t - t^2 - 3t^3}.$$ 

(b) Show that

$$a_n = \sum_{j \geq 0} \sum_{m \geq 0} 4^j \left[ \binom{n-2m}{j} m + \binom{n-2m-1}{j} \binom{m-1}{j} + 2 \binom{n-2m}{j} \binom{m-1}{j} \right].$$ 

Here, $\binom{k}{l}$ = 0 if $k < l$. 

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PROBLEMS AND SOLUTIONS

Edited by Gerald A. Edgar, Doug Hensley, Douglas B. West

Proposed problems and solutions should be sent in duplicate to the MONTHLY problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before September 30, 2014. Additional information, such as generalizations and references, is welcome. The problem number and the solver’s name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.
11777. Proposed by Marian Dincă, Bucharest, Romania. Let \( x_1, \ldots, x_n \) be real numbers such that \( \prod_{k=1}^{n} x_k = 1 \). Prove that
\[
\sum_{k=1}^{n} \frac{x_k^2}{x_k^2 - 2x_k \cos(2\pi/n) + 1} \geq 1.
\]

11778. Proposed by Li Zhou, Polk State College, Winter Haven, FL. Let \( x, y, z \) be positive real numbers such that \( x + y + z = \pi/2 \). Let \( f(x, y, z) = 1/(\tan^2 x + 4\tan^2 y + 9\tan^2 z) \). Prove that
\[
f(x, y, z) + f(y, z, x) + f(z, x, y) \leq \frac{9}{14}(\tan^2 x + \tan^2 y + \tan^2 z).
\]

11779. Proposed by Michel Bataille, Rouen, France. Let \( M, A, B, C, \) and \( D \) be distinct points (in any order) on a circle \( \Gamma \) with center \( O \). Let the medians through \( M \) of triangles \( MAB \) and \( MCD \) cross lines \( AB \) and \( CD \) at \( P \) and \( Q \), respectively, and meet \( \Gamma \) again at \( E \) and \( F \), respectively. Let \( K \) be the intersection of \( AF \) with \( DE \), and let \( L \) be the intersection of \( BF \) with \( CE \). Let \( U \) and \( V \) be the orthogonal projections of \( C \) onto \( MA \) and \( D \) onto \( MB \), respectively, and assume \( U \neq A \) and \( V \neq B \). Prove that \( A, B, U, \) and \( V \) are concyclic if and only if \( O, K, \) and \( L \) are collinear.

11780. Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Tudorel Lupu, Decebal High School, Constanța, Romania. Let \( f \) be a positive-valued, concave function on \([0, 1]\). Prove that
\[
\frac{3}{4} \left( \int_{0}^{1} f(x) \, dx \right)^2 \leq \frac{1}{8} + \int_{0}^{1} f^3(x) \, dx.
\]

11781. Proposed by Roberto Tauraso, Università di Roma “Tor Vergata”, Rome, Italy. For \( n \geq 2 \), call a positive integer \( n \)-smooth if none of its prime factors is larger than \( n \). Let \( S_n \) be the set of all \( n \)-smooth positive integers. Let \( C \) be a finite, nonempty set of nonnegative integers, and let \( a \) and \( d \) be positive integers. Let \( M \) be the set of all positive integers of the form \( m = \sum_{k=1}^{d} c_k s_k \), where \( c_k \in C \) and \( s_k \in S_n \) for \( k = 1, \ldots, d \). Prove that there are infinitely many primes \( p \) such that \( p^a \notin M \).

SOLUTIONS

Integrals with Bernoulli Numbers

11644 [2012, 426]. Proposed by Albert Stadler, Herrliberg, Switzerland. Let \( n \) be a nonnegative integer, and let \( B_j \) be the \( j \)th Bernoulli number, defined for \( j \geq 0 \) by
\[
B_j = \frac{1}{j!} \int_{0}^{1} (\ln(1+t))^j \, dt.
\]
If \( f \) and we compute \( f \) the integral converges absolutely for \( \Re s \) contained in \( \Re s \) in the whole complex plane. Also, the residues of \( f \) Note that (1) represents the analytic continuation of \( f \) constant.

Prove that \( I \) in a neighborhood of \( x = 0 \). Define

\[
I_n = \int_0^\infty \left( \frac{1}{x^n(e^x - 1)} - \left( \frac{1}{x^n} + \sum_{k=0}^n B_k \frac{x^{k-n-1}}{k!} \right) e^{-x} \right) dx.
\]

Prove that \( I_0 = \gamma - 1 \), that \( I_1 = 1 - (1/2) \log(2\pi) \), and that for \( n \geq 1 \),

\[
I_{2n} = (\log(2\pi) + \gamma) \frac{B_{2n}}{(2n)!} + (-1)^n \frac{2\zeta'(2n)}{(2\pi)^{2n}} \sum_{k=0}^{n-1} \frac{B_{2k}}{(2k)!} H_{2n-1} - \frac{1}{2(2n-1)!} H_{2n-1} - \sum_{k=0}^{n} \frac{B_{2k}}{(2k)!} \frac{H_{2n+1-2k}}{(2n + 1 - 2k)!},
\]

and that for \( n \geq 1 \),

\[
I_{2n+1} = (-1)^n \frac{\zeta(2n+1)}{2(2\pi)^{2n}} - \frac{1}{2(2n)!} H_{2n} + \sum_{k=0}^{n} \frac{B_{2k}}{(2k)!} \frac{H_{2n+1-2k}}{(2n + 1 - 2k)!}.
\]

Here, \( H_n \) denotes \( \sum_{k=1}^{n} 1/k \), \( \zeta \) denotes the Riemann zeta function, and \( \gamma \) is Euler’s constant.

Solution by the proposer. Note that

\[
\frac{e^x}{x^n(e^x - 1)} = \frac{1}{x^n} + \sum_{k=0}^n B_k \frac{x^{k-n-1}}{k!} + O(1)
\]

in a neighborhood of \( x = 0 \). Define

\[
f_n(s) = \int_0^\infty x^{s-1} \left( \frac{1}{x^n(e^x - 1)} - \left( \frac{1}{x^n} + \sum_{k=0}^n B_k \frac{x^{k-n-1}}{k!} \right) e^{-x} \right) dx.
\]

The integral converges absolutely for \( \Re s > 0 \) and uniformly in every compact subset contained in \( \Re s \geq \epsilon > 0 \). Therefore, \( f_n(s) \) is analytic in \( \Re s > 0 \). Thus \( I_n = f_n(1) \), and we compute \( f_n(1) \).

If \( \Re s > n \), then

\[
f_n(s) = \Gamma(s-n)\zeta(s-n) - \Gamma(s-n) - \sum_{k=0}^n \frac{B_k}{k!} \Gamma(s+k-n-1) \quad (1)
\]

Note that (1) represents the analytic continuation of \( f_n(s) \) as a meromorphic function in the whole complex plane. Also, the residues of \( f_n \) at \( s \in \{1, \ldots, n\} \) all vanish.

We now take note of some well-known facts about the gamma and zeta functions. If \( m \) is a nonnegative integer, then in a neighborhood of \( s = 1 \) we have

\[
\Gamma(s-m) = \frac{\Gamma(s)}{(s-1)(s-2) \cdots (s-m)} = \frac{(-1)^{m-1}}{(m-1)!} \left( \frac{1}{s-1} - \gamma + H_{m-1} + O(s-1) \right) \quad (2)
\]

By considering the residue of \( f_n \) at \( 1 \), we have

\[
0 = \frac{(-1)^{n-1}}{(n-1)!} \zeta(1-n) - \frac{(-1)^{n-1}}{(n-1)!} - \sum_{k=0}^{n} \frac{B_k}{k!} \frac{(-1)^{n-k}}{(n-k)!}.
\]
For the last equality we used
\[ xe^{-x} = \left( \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \right) (1 - e^{-x}) = \left( \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \right) \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k!} \right) \]
and compared coefficients of \( x^n \). Therefore,
\[ \zeta(1 - n) = (-1)^{n-1} \frac{B_n}{n}. \] (3)

Using the functional equation
\[ \zeta(1 - s) = 2(2\pi)^{-s} \cos \frac{\pi s}{2} \Gamma(s) \zeta(s), \] (4)
we get
\[ 2^{1-n} \pi^{-n} \cos \frac{\pi n}{2} \Gamma(n) \zeta(n) = (-1)^{n-1} \frac{B_n}{n} \]
and
\[ \zeta(2n) = (-1)^{n-1} \frac{B_{2n}}{2(2n)!} (2\pi)^{2n}. \]

Now
\[
I_0 = f_0(1) = \lim_{s \to 1} \left( \Gamma(s) \zeta(s) - \Gamma(s) - \Gamma(s-1) \right) = -1 + \lim_{s \to 1} \frac{\Gamma(s)(s-1) \zeta(s) - \Gamma(s)}{s-1} \\
= -1 + \lim_{s \to 1} \frac{1}{s-1} \left( [1 - \gamma(s-1) + O((s-1)^2)] [1 + \gamma(s-1) + O((s-1)^2)] - [1 - \gamma(s-1) + O((s-1)^2)] \right) = \gamma - 1.
\]

For \( n \geq 1 \),
\[
I_n = \int_0^\infty \left( \frac{1}{x^n (e^x - 1)} - \frac{1}{x^n} + \sum_{k=0}^n B_k \frac{x^{k-n-1}}{k!} \right) e^{-x} \, dx = f_n(1)
\]
\[
= \lim_{s \to 1} \left( \Gamma(s-n) \zeta(s-n) - \Gamma(s-n) - \sum_{k=0}^n B_k \frac{\Gamma(s+k-n-1)}{k!} \right)
\]
\[
= \frac{(-1)^{n-1}}{(n-1)!} \left( -\gamma + H_{n-1} \right) \zeta(1-n) + \frac{(-1)^{n-1}}{(n-1)!} \zeta'(1-n)
\]
\[
- \frac{(-1)^{n-1}}{(n-1)!} \left( -\gamma + H_{n-1} \right) - \sum_{k=0}^n B_k \frac{\Gamma(s+k-n-1)}{k!} \cdot \frac{(-1)^{n-k}}{(n-k)!} \zeta'(1-n)
\]
\[
= \frac{(-1)^{n-1}}{(n-1)!} \left( \zeta(1-n) - 1 \right) + \frac{(-1)^{n-1}}{(n-1)!} \zeta'(1-n) - \sum_{k=0}^{n-1} B_k \frac{\Gamma(s+k-n-1)}{k!} \cdot \frac{(-1)^{n-k}}{(n-k)!} H_{n-k}
\]
\[
= \frac{B_{n}}{n!} H_{n-1} - \frac{(-1)^{n-1}}{(n-1)!} H_{n-1} + \frac{(-1)^{n-1}}{(n-1)!} \zeta'(1-n) - \sum_{k=0}^{n-1} B_k \frac{\Gamma(s+k-n-1)}{k!} \cdot \frac{(-1)^{n-k}}{(n-k)!} H_{n-k}.
\]
where we have used (2) and the fact that the residue at 1 of \( f_n \) is zero. To get from here to the required formulas, we will need to relate the values of \( \zeta' \) at negative integers to values of \( \zeta \) and \( \zeta' \) at positive integers.

We have \( \zeta(0) = -1/2 \). From (4) we deduce

\[
-\frac{\zeta'(1 - s)}{\zeta(1 - s)} = -\log(2\pi) - \frac{\pi}{2} \tan \frac{\pi s}{2} + \frac{\Gamma''(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}.
\]

In a neighborhood of \( s = 1 \),

\[
\frac{\pi}{2} \tan \frac{\pi s}{2} = -\frac{1}{s - 1} + O(s - 1), \quad \frac{\Gamma'(s)}{\Gamma(s)} = -\gamma + O(s - 1),
\]

and

\[
\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s - 1} + \gamma + O(s - 1),
\]

so

\[
-\frac{\zeta'(0)}{\zeta(0)} = -\log(2\pi), \quad \zeta'(0) = -\frac{1}{2} \log(2\pi).
\]

We have

\[
\frac{\Gamma'(s + n)}{\Gamma(s + n)} = \frac{1}{s + n - 1} + \frac{1}{s + n - 2} + \cdots + \frac{1}{s + 1} + \frac{\Gamma'(s + 1)}{\Gamma(s + 1)}.
\]

Thus,

\[
\frac{\Gamma'(n)}{\Gamma(n)} = H_{n-1} - \gamma, \quad \Gamma'(n) = (n - 1)! (H_{n-1} - \gamma).
\]

From (4), we deduce

\[
-\zeta'(1 - s) = -2 \log(2\pi)(2\pi)^{-s} \cos \frac{\pi s}{2} \Gamma(s) \zeta(s) - 2(2\pi)^{-s} \frac{\pi}{2} \sin \frac{\pi s}{2} \Gamma(s) \zeta(s)
\]

\[
+ 2(2\pi)^{-s} \cos \frac{\pi s}{2} \Gamma'(s) \zeta(s) + 2(2\pi)^{-s} \cos \frac{\pi}{2} \Gamma'(s) \zeta'(s).
\]

For \( n \geq 1 \), let \( Z_n = \zeta'(1 - n)(2\pi)^n / 2(n - 1)! \). We then have

\[
Z_n = \log(2\pi) \cos \frac{\pi n}{2} \zeta(n) + \frac{\pi}{2} \sin \frac{\pi n}{2} \zeta(n)
\]

\[
- \cos \frac{\pi n}{2} (H_{n-1} - \gamma) \zeta(n) - \cos \frac{\pi n}{2} \zeta'(n).
\]

Thus for odd \( n \), \( Z_n = \frac{\pi}{2} (-1)^{(n-1)/2} \zeta(n) \), while for even \( n \),

\[
Z_n = (-1)^{n/2} \left[ (\log(2\pi) - H_{n-1} + \gamma) \zeta(n) - \zeta'(n) \right]
\]

\[
= (-\log(2\pi) + H_{n-1} - \gamma) \frac{B_n(2\pi)^n}{2(n!)} - (-1)^{n/2} \zeta'(n).
\]

We thus conclude:

\[
I_0 = \gamma - 1, \quad I_1 = f_1(1) = \zeta'(0) + B_0 H_1 = 1 - \frac{1}{2} \log(2\pi),
\]

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and for \( n \geq 1 \), using the fact that \( B_1 = -1/2 \) while \( B_{2k+1} = 0 \) for \( k > 0 \),

\[
I_{2n} = \frac{1}{2(2n-1)!} H_{2n-1} + (\log(2\pi) + \gamma) \frac{B_{2n}}{(2n)!} + \frac{(-1)^n 2\zeta'(2n)}{(2\pi)^{2n}} - \sum_{k=0}^{n-1} \frac{B_{2k}}{(2k)!} \cdot \frac{H_{2n-2k}}{(2n-2k)!},
\]

\[
I_{2n+1} = -\frac{1}{2(2n)!} H_{2n} + \frac{(-1)^n \zeta(2n+1)}{2(2\pi)^{2n}} + \sum_{k=0}^{n} \frac{B_{2k}}{(2k)!} \cdot \frac{H_{2n+1-2k}}{(2n+1-2k)!}.
\]

Also solved by B. Burdick.

### An \( l^p \) Inequality

**Theorem** (Hardy–Littlewood–Pólya, 1952). Proposed by Grahame Bennett, Indiana University, Bloomington, IN. Let \( p \) be real with \( p > 1 \). Let \( (x_0, x_1, \ldots) \) be a sequence of nonnegative real numbers. Prove that

\[
\sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{x_k}{j+k+1} \right)^p < \infty \implies \sum_{j=0}^{\infty} \left( \frac{1}{j+1} \sum_{k=0}^{j} x_k \right)^p < \infty.
\]

**Solution by Oliver Geupel, Brühl, NRW, Germany.** For every nonnegative integer \( j \), since \( x_j > 0 \), we have

\[
\frac{1}{j+1} \sum_{k=0}^{j} x_k \leq \frac{2j+1}{j+1} \sum_{k=0}^{j} \frac{x_k}{j+k+1} \leq 2 \sum_{k=0}^{\infty} \frac{x_k}{j+k+1}.
\]

If \( p > 0 \), then \( x^p \) strictly increases with \( x \) on the interval \([0, \infty)\). Thus, raising both sides of this inequality to the \( p \)th power and summing both sides over \( j \) yields

\[
\sum_{j=0}^{\infty} \left( \frac{1}{j+1} \sum_{k=0}^{j} x_k \right)^p \leq 2^p \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{x_k}{j+k+1} \right)^p.
\]

The proof also shows that the restriction on \( p \) can be relaxed to \( p > 0 \).

**Editorial comment.** Kenneth F. Anderson remarked that, conversely, since \((a+b)^p \leq 2^p(a^p + b^p)\) for \( a, b \geq 0 \), it follows that

\[
\sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{x_k}{j+k+1} \right)^p \leq 2^p \left[ \sum_{j=0}^{\infty} \left( \frac{1}{j+1} \sum_{k=0}^{j} x_k \right)^p + \sum_{j=0}^{\infty} \left( \sum_{k=j}^{\infty} x_k \right)^p \right].
\]

The convergence of the two series on the right-hand side implies convergence of \( \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} x_k/(j+k+1) \right)^p \). See Hardy’s discussion of Hilbert’s Double Series Theorem (Hardy–Littlewood–Pólya, Inequalities, Cambridge University Press, 1967, Ch. 9).

Also solved by K. F. Andersen (Canada), R. Bagby, P. P. Dályay (Hungary), E. A. Herman, F. Holland (Ireland), B. Karaivanov, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), P. Perfetti (Italy), M. A. Prasad (India), A. Stenger, R. Stong, R. Tauraso (Italy), T. Viteam (Chile), and the proposer.
A Double Integral

11650 [2012, 522]. Proposed by Michael Becker, University of South Carolina at Sumter, Sumter, SC. Evaluate

\[ \int_{x=0}^{\infty} \int_{y=x}^{\infty} e^{-(x-y)^2} \sin^2(x^2 + y^2) \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \, dx. \]

Solution by Jan A. Van Casteren, University of Antwerp, Antwerp, Belgium. As preparation, we evaluate the following integral for \( \sigma > 0 \):

\[ \int_{0}^{\infty} e^{-2\sigma \rho} \frac{\sin^2 \rho}{\rho} \, d\rho = \int_{0}^{\infty} e^{-\sigma \rho/2} \frac{\sin^2 (\rho/2)}{\rho} \, d\rho = \frac{1}{2} \int_{0}^{\infty} e^{-\rho} d\tau (1 - \cos \rho) d\rho \]

\[ = \frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\rho} (1 - \cos \rho) d\rho \, d\tau = \frac{1}{2} \int_{0}^{\infty} \left( \frac{1}{\tau} - \frac{\tau}{1 + \tau^2} \right) d\tau \]

\[ = \frac{1}{2} \log \left( \frac{(1 + \sigma^2)^{1/2}}{\sigma} \right). \]

Now for the integral \( J \) of the problem: passing first to polar coordinates via \( x = r \cos \varphi, \ y = r \sin \varphi \), we compute

\[ J = \int_{0}^{\infty} \int_{x}^{\infty} e^{-(x-y)^2} \sin^2(x^2 + y^2) \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \, dx \]

\[ = \int_{0}^{\infty} \int_{\pi/4}^{\pi/2} e^{-r^2 + 2r^2 \sin \varphi \cos \varphi} \sin^2(r^2) \frac{\cos^2 \varphi - \sin^2 \varphi}{r} \, d\varphi \, dr \]

\[ = \int_{0}^{\infty} \int_{\pi/4}^{\pi/2} e^{-r^2 + r^2 \sin 2\varphi} \cos 2\varphi \frac{\sin^2(r^2)}{r} \, d\varphi \, dr \]

\[ = -\frac{1}{2} \int_{0}^{\infty} \frac{1 - e^{-r^2}}{r^2} \, r^2 \, dr \quad \text{(substitute } \rho = r^2) \]

\[ = -\frac{1}{4} \int_{0}^{\infty} \frac{1 - e^{-\rho}}{\rho} \sin^2 \rho \, d\rho = -\frac{1}{2} \int_{0}^{1/2} \int_{0}^{\infty} e^{-2\sigma \rho} \frac{\sin^2 \rho}{\rho} \, d\rho \, d\sigma \]

\[ = -\frac{1}{4} \int_{0}^{1/2} \log \left( \frac{1 + (\sigma^2)^{1/2}}{\sigma} \right) d\sigma \quad \text{(integrate by parts)} \]

\[ = -\frac{1}{4} \left( \frac{1}{2} \log \left( \frac{1 + (1/2)^{1/2}}{1/2} \right) + \arctan \frac{1}{2} \right) = -\frac{1}{16} \log 5 - \frac{1}{4} \arctan \frac{1}{2}. \]


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A Binomial Determinant

11652 [2012, 522–523]. Proposed by Ajai Choudhry, Foreign Service Institute, New Delhi, India. For \(a, b, c, d \in \mathbb{R}\), and for nonnegative integers \(i, j, \text{and } n\), let

\[t_{i,j} = \sum_{s=0}^{j} \binom{n-i}{j-s} a^{n-i-j+s} b^{j-s} c^{i-s} d^s.\]

Let \(T(a, b, c, d, n)\) be the \((n+1)\)-by-\((n+1)\) matrix with \((i, j)\)-entry given by \(t_{i,j}\), for \(i, j \in \{0, \ldots, n\}\). Show that det \(T(a, b, c, d, n) = (ad - bc)^{n(n+1)/2}\).

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let \(E\) denote the vector space \(\mathbb{R}_n[x]\) of real polynomials with degree at most \(n\), and let \(B\) denote the canonical basis \(\{1, x, x^2, \ldots\}\) of \(E\). Consider the linear transformations \(V\) and \(T_{b,c}\) from \(E\) to \(E\) defined by \(V(P(x)) = x^n P(1/x)\) and \(T_{b,c}(P(x)) = P(\lambda x + \mu)\), where \((\lambda, \mu) \in \mathbb{R}^2\).

For a linear transformation \(T\) from \(E\) to \(E\), let det \((T)\) denote the determinant of the matrix of \(T\) with respect to \(B\). Since the matrices of \(V\) and \(T_{b,c}\) with respect to \(B\) are

\[
\begin{pmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 \\
\vdots & \cdots & \vdots & \vdots \\
0 & 1 & 0 & \cdots \\
1 & 0 & \cdots & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & \mu & * & \cdots & * \\
0 & \lambda & * & \cdots & * \\
0 & 0 & \lambda^2 & \cdots & * \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \lambda^n
\end{pmatrix},
\]

we obtain det \((V) = (-1)^{n(n+1)/2}\) and det \((T_{b,c}) = \lambda^{n(n+1)/2}\).

Now consider \((a, b, c, d) \in \mathbb{R}^4\) with \(b \neq 0\), and let \(U\) be the linear transformation defined by \(U = T_{b,a} \circ V \circ T_{c-ad/b,d/b}\). We have

\[\text{det}(U) = \text{det}(T_{b,a}) \text{det}(V) \text{det}(T_{c-ad/b,d/b}) = (ad - bc)^{n(n+1)/2}. \quad (*)\]

On the other hand, for \(0 \leq i \leq n\),

\[U(x^i) = (a + bx)^{n-i}(c + dx)^i = \sum_{j=0}^{n} \left( \sum_{s \geq 0} \binom{n-i}{j-s} a^{n-i-j+s} b^{j-s} c^{i-s} d^s \right) x^j = \sum_{j=0}^{n} t_{i,j} x^j.\]

Thus, the matrix of \(U\) with respect to \(B\) is the transpose of the matrix \(T(a, b, c, d, n)\).

Using \((*)\), we obtain

\[\text{det}(T(a, b, c, d, n)) = \text{det}(U) = (ad - bc)^{n(n+1)/2}.\]

The case \(b = 0\) follows by continuity.