Problem. (Pitt Prelim, 2009)
(a) If $a > 1$ and $k > 1$, then $\sum_{n=2}^{\infty} \frac{(\log n)^k}{n^a} < \infty$.
(b) The function $g(x) = \sum_{n=1}^{\infty} \frac{1}{nx}$, $x > 1$, is infinitely differentiable in $(1, \infty)$.

Solution.

(a) Let $p$ be a fixed number satisfying $1 < p < a$. Set $\epsilon = a - p$. Now, using the fact that $(\log n)^k < n^\epsilon$ for all large $n$, and the fact that the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, we obtain:

$$\sum_{n=2}^{\infty} \frac{(\log n)^k}{n^a} = \left(\sum_{n=2}^{\infty} \frac{1}{n^p}\right) \left(\sum_{n=2}^{\infty} \frac{\log n}{n^\epsilon}\right) \leq \infty$$

(b) For this, we use the following theorem:

**Theorem.**

If $f_n \in C^1(a, b)$ for all $n \in \mathbb{N}$,

$$\lim_{N \to \infty} \sum_{n=1}^{N} f_n(x) = f(x) \text{ pointwise for } x \in (a, b),$$

and

$$\lim_{N \to \infty} \sum_{n=1}^{N} f'_n(x) = g(x) \text{ uniformly in } x \in (a, b),$$

then $f \in C^1(a, b)$ and $f(x) = g(x)$. 
To see that the series has derivatives of all continuous orders, we first note that the summand \( \frac{1}{n^x} \) is smooth and for each integer \( k \geq 0 \),
\[
\left( \frac{1}{n^x} \right)^{(k)} = (e^{-x \ln n})^{(k)} = (-\ln n)^{k} \cdot e^{-x \ln n} = (-\ln n)^{k} \cdot \frac{x^k}{n^x}.
\]

Now, let \( a > 1 \) be arbitrary. Since, we have
\[
\frac{(-\ln n)^{k}}{n^x} \leq \frac{(\ln n)^{k}}{n^a}, \quad \forall x \geq a,
\]
by the Weierstrass \( n \)-test, it follows that
\[
\sum_{n=2}^{\infty} \frac{(\ln n)^{k}}{n^x}
\]
converges uniformly in \( (a, \infty) \). Then, according to the above theorem, we can inductively show that \( g \) is \( k \)-th differentiable in \( (a, \infty) \) for \( k = 0, 1, 2, \ldots \) and
\[
g(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad g'(x) = \sum_{n=1}^{\infty} \frac{-\ln n}{n^x}, \quad g^{(k)}(x) = \sum_{n=1}^{\infty} \frac{(-\ln n)^{k}}{n^x},
\]

Since \( a \) was arbitrary, \( g \in C^{\infty}(a, \infty) = C^{\infty}(1, \infty) \) and we are done. \( \square \)
Problem. (Pitt Prelim, 2015)

Let \( f_n : X \to \mathbb{R}, n = 1, 2, \ldots \), be a sequence of continuous functions on a metric space \( X \) such that the series \( \sum_{n=1}^{\infty} f_n(x) \) converges for all \( x \in X \), and

\[
\sup_{x \in X} \left( \sum_{n=1}^{\infty} f_n^2(x) \right)^{\frac{1}{2}} < \infty
\]

Prove that if a sequence of real numbers \( \alpha_n, n = 1, 2, \ldots \) satisfies \( \sum_{n=1}^{\infty} \alpha_n^2 < \infty \), then the series \( \sum_{n=1}^{\infty} \alpha_n f_n(x) \) converges everywhere to a continuous function.

Solution.

It suffices to show the U.C. of \( \sum_{n=1}^{\infty} \alpha_n f_n(x) \).

To this end it suffices to verify Cauchy condition.

\[ \exists \varepsilon > 0, \exists m_0, \forall M > m_0, \forall n \in \mathbb{N}, \sup_{x \in X} \left| \sum_{k=n}^{M} \alpha_k f_k(x) \right| \leq \varepsilon. \]

Let \( \varepsilon > 0 \) be such that

\[
\sup_{x \in X} \left( \sum_{n=1}^{\infty} f_n^2(x) \right)^{\frac{1}{2}} \leq A.
\]

Since \( \sum_{n=1}^{\infty} \alpha_n^2 < \infty \), \( \forall \varepsilon > 0 \), there is no such that

\[ M > M_0 \text{, we have } \sum_{n=M}^{\infty} \alpha_n^2 \leq \varepsilon^2 A^2. \]
Then the Cauchy-Schwarz inequality yields

\[ \left| \sum_{n=N}^{M} a_n f_n(x) \right| \leq \left( \sum_{n=N}^{M} a_n^2 \right)^{1/2} \left( \sum_{n=N}^{M} f_n(x)^2 \right)^{1/2} \]

\[ \leq (\varepsilon^2 A^2)^{1/2} A = \varepsilon. \]

Since this is true for any \( x \in X, \)

\[ \sup_{x \in X} \left| \sum_{n=N}^{M} a_n f_n(x) \right| \leq \varepsilon, \text{ for all } \]

\[ M \geq N \geq n_0. \]
Problem. (Pitt Prelim, 2006)

(a) Give sufficient conditions under which the series \( \sum_{n=1}^{\infty} f_n(x) \) can be differentiated term by term on a bounded interval \( I \subseteq \mathbb{R} \).

(b) Can we differentiate \( \sum_{n=1}^{\infty} \arctan \frac{x}{n^2} \) term by term on \( \mathbb{R} \)?

Solution.

(a) The following three conditions are sufficient for the series \( \sum_{n=1}^{\infty} f_n(x) \) to be differentiated term by term:

1. \( f_n \) is continuous on \( I \).
2. \( \sum_{n=1}^{\infty} f_n(x) \) converges at one point on \( I \).
3. \( \sum_{n=1}^{\infty} f_n'(x) \) converges uniformly on \( I \).

(b) Yes, we can differentiate term by term! For arbitrary \( M > 0 \), let \( I = [-M, M] \) and let \( f_n(x) = \arctan \frac{x}{n^2} \). Then \( f_n \) is differentiable on \( I \). Also, let \( x = 0 \), we have \( \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \arctan \frac{0}{n^2} = 0 \), so \( \sum_{n=1}^{\infty} f_n(x) \) converges at \( x = 0 \). Further, we have

\[
0 \leq f_n'(x) = \frac{1}{n^2 + x^2} \leq \frac{1}{n^2}, \quad \forall x \in \mathbb{R},
\]

by \( M \)-test, we are done.
Problem. (Ohio State Prelim, 2005).
Prove that
\[ f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \log(1 + \frac{x}{n}) \]
is defined and differentiable on the open interval \(-1 < x < \infty\).

Solution.
If \( f \) is defined correctly \( \iff \) \( 1 + \frac{x}{n} \geq 0 \implies \frac{x}{n} > -1 \)
which is evident.

Step 1. The series \( \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \log(1 + \frac{x}{n}) \) is uniformly convergent.

Denote by \( f_n(x) = (-1)^{n+1} \cdot \log(1 + \frac{x}{n}) \). Clearly \( f_n \) is differentiable and moreover
\[ |f_n(x)| = \left| (-1)^{n+1} \cdot \frac{1}{1 + \frac{x}{n}} \right| = \frac{1}{\frac{1}{n} + \frac{x}{n}} \leq \left| \frac{x}{n} \right| \leq \frac{M}{n} \]

Unfortunately, the series \( \sum_{n=1}^{\infty} \frac{M}{n} \) is divergent and thus, we cannot apply Weierstrass M-test.

We will prove that the series \( \sum_{n=1}^{\infty} f_n(x) \) is convergent.
We have: \( x \leq M < \infty \), so
\[ \left| (-1)^{n+1} \cdot \log(1 + \frac{x}{n}) \right| \leq (-1)^{n+1} \cdot \log(1 + \frac{M}{n}) \]
Moreover, the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \log(1 + \frac{M}{n}) \) converges by Leibniz criterion, so \( \sum f_n(x) \) is convergent.

Step 2. The series is \( \sum_{n=1}^{\infty} f_n(x) \) is uniformly convergent.
We have: \( f_n'(x) = (-1)^{n+1} \cdot \frac{1}{x+n} = (-1)^{n+1} \cdot \frac{1}{x+n} \).

The series \( \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{x+n} \) is uniformly convergent, but we cannot apply the \( p \)-test here, but Dirichlet's test, namely:

**Theorem (Dirichlet)**

Let \( (f_n)_{n \geq 1}, (g_n)_{n \geq 1} : A \subseteq \mathbb{R} \rightarrow \mathbb{R} \) two sequences of functions such that:

1. \( f_n \xrightarrow{m} 0 \) and \( (f_n)_{n \geq 1} \) is decreasing;
2. \( g_n = \sum_{k=1}^{n} g_k \) is bounded, i.e.

\[ (\exists) M > 0 \text{ such that } (\forall) n \in \mathbb{N}^*, \forall x \in A, \quad |g_1(x) + \ldots + g_n(x)| \leq M. \]

\[ \Rightarrow \sum_{n=1}^{\infty} f_n(x) g_n(x) \text{ converges uniformly on } A. \]

In our case, take: \( f_n(x) = \frac{1}{x+n} \xrightarrow{m} 0 \) and \( f_n \) is decreasing and \( g_n = \sum_{k=1}^{n} (-1)^{k+1} \) is bounded, so by the Dirichlet criterion, it follows that \( \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{x+n} \) is uniformly convergent.

Now, it follows that \( \sum_{n=1}^{\infty} f_n(x) \) is uniformly convergent and moreover \( f'(x) = \sum_{n=1}^{\infty} f_n'(x) \) is differentiable and we can differentiate term by term, i.e. \( \left( \sum_{n=1}^{\infty} f_n(x) \right)' = \sum_{n=1}^{\infty} f_n'(x) \).
REMARK.

Here we can take into account the following important theorems:

THEOREM 1.

Let $I \subseteq \mathbb{R}$ be a bounded interval and $\sum_{n=1}^{\infty} f_n(x)$ is a series of differentiable functions on $I$. If $\sum_{n=1}^{\infty} f_n(x)$ is convergent and $\sum_{n=1}^{\infty} f'_n(x)$ is uniformly convergent to a function $g$, then:

(i) $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $I$ to $f$;

(ii) $f$ is differentiable, $f(x) = g(x)$, i.e.,

$$\left( \sum_{n=1}^{\infty} f_n(x) \right)' = \sum_{n=1}^{\infty} f'_n(x).$$

THEOREM 2.

Let $I \subseteq \mathbb{R}$ be a bounded interval and $\sum_{n=1}^{\infty} f_n(x)$ is a series of differentiable functions such that $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent to a function $f: I \rightarrow \mathbb{R}$ and $\sum_{n=1}^{\infty} f'_n(x)$ converges uniformly to $g: I \rightarrow \mathbb{R}$. Then $f$ is differentiable and $f(x) = g(x)$, i.e.,

$$\left( \sum_{n=1}^{\infty} f_n(x) \right)' = \sum_{n=1}^{\infty} f'_n(x).$$
Problem. (Ohio State University Qualifying Exam) Consider the sequence of functions \( f_n : [0,1] \to \mathbb{R} \) which is given by the recursion:

\[ f_1 \equiv 0 \quad \text{and} \quad f_{n+1}(x) = f_n(x) + \frac{1}{2} [x - f_n(x)], \quad x \in [0,1], n \geq 1 \]

Show that the sequence \( (f_n)_{n \geq 1} \) converges uniformly to \( f(x) = \sqrt{x} \).

Solution. First, we show by induction that \( f_n(x) \leq \sqrt{x} \), for each \( n \geq 1 \), and for all \( x \in [0,1] \). For \( n = 1 \), we have \( f_1 \equiv 0 \leq \sqrt{x} \), obvious! Now, assume that \( f_m(x) \leq \sqrt{x} \), for \( m \geq 1 \) and \( x \in [0,1] \). Then, we have

\[
\begin{align*}
\frac{f_{n+1}(x) - f_n(x)}{\sqrt{x} - f_n(x)} &= \frac{f_n(x) + \frac{1}{2} [x - f_n(x)] - f(x)}{\sqrt{x} - f_n(x)} \\
&= \frac{(f_n(x) - f(x)) + \frac{1}{2} [x - f_n(x)]}{\sqrt{x} - f_n(x)} \\
&\leq 0 \quad \text{(Induction hypothesis)}
\end{align*}
\]

\[
\leq \sqrt{x} - f_n(x) \leq \sqrt{x} - \sqrt{x} = 0,
\]

as desired. Therefore, we have: \( [0,1] f_n(x) \leq \sqrt{x} \) for \( n \geq 1 \), and for any \( x \in [0,1] \). Since \( f_m(x) \leq \sqrt{x} \), for \( m \geq 1 \) and \( x \in [0,1] \), we have that \( f_n(x) \) is increasing and bounded by \( \sqrt{x} \) for each \( x \in [0,1] \). Hence, the pointwise limit exists.

Now, for each \( x \), let \( l(x) = \lim_{n \to \infty} f_n(x) \). Passing to the limit in the recursion \( f_{n+1} \), we get

\[
l(x) = l(x) + \frac{x - l(x)^2}{2} \quad \Rightarrow \quad l(x) = \sqrt{x},
\]
Since \( f_n(x) \) is continuous and it is converging monotonically to a function \( f(x) = \sqrt{x} \) on a compact set, by Dini's theorem, \( f_n(x) \rightarrow f(x) = \sqrt{x} \). □

What does this theorem say?
Problem. (Rudin, chapter 7)

prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of $x$.

Solution.

We can write

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^2 + n}{n^2} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^2}{n^2} + \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$$

Both series converge uniformly and absolutely on any bounded interval.

On the interval $[a, b]$ by the $n$-test (with the sequence $M_n = \frac{M^2}{n^2}$, where $M = \max \{ |a|, |b| \}$).

The second series (series of real numbers).

The series does not converge absolutely since each term (the absolute value of each term) is at least $\frac{1}{n}$ for any $x$. $\square$
Problem. (Berkeley Prelim Exam).

Let \((g_n)_n\) be a sequence of twice differentiable functions on \([-1,1]\) such that for all \(n\), \(g_n(0) = 0\) and \(g'_n(0) = 0\). Suppose that \(\left| g''_n(x) \right| \leq 1\) for all \(n, x\).

Prove that there is a subsequence of \((g_n)_n\) which converges uniformly on \([-1,1]\).

**Solution.**

We have to prove that \((g_n)_n\) is bounded and equicontinuous for all \(n, x\). We have:

\[
g_n(x) = g_n(0) + \frac{x^2}{2!} g_n'(0) + \frac{x^2}{2} g''(0)
\]

\[
= \frac{x^2}{2} g''(0) \leq \frac{1}{2} g''(0)
\]

\[
\Rightarrow \left| g_n(x) \right| \leq \frac{1}{2} \left| g''(0) \right| \leq \frac{1}{2} \Rightarrow g_n \text{ is bounded.}
\]

Now, we prove that it is equicontinuous. We have:

\[
\left| g_n(x) - g_n(y) \right| = \left| g_n'(x) \cdot (x-y) \right|, \text{ for all } x, y \in [-1,1]
\]

and for some \(\varepsilon\) between \(x\) and \(y\).

We have to prove that \(g_n\) is \(\varepsilon\)-continuous.

For this, we have:

\[
\left| g_n'(x) \right| = \left| g_n'(x) - g_n'(y) \right| \leq \left| g_n''(\xi) (x-y) \right|
\]

\[
\leq 1, \text{ for } \frac{\varepsilon}{2} \in [0,1],
\]

and we are done. \(\square\)
Problem. (University of Pittsburgh Prelim, 2013) 
prove that \( \sum_{n=1}^{\infty} \frac{x^2}{\sqrt{n(n^2+x^3)}} \) is uniformly convergent
on \([0,\infty)\) if \(d = 2\), but not uniformly convergent on
\([0,1)\) if \(d = 3\).

Solution.
Let's prove that the series \( \sum_{n=1}^{\infty} \frac{x^2}{\sqrt{n(n^2+x^3)}} \) is U.C.
We will give two methods for this:

Method 1. We split our problem into two parts:

On the interval \([0,6]\), we have:

\[ \left| \frac{x^2}{\sqrt{n(n^2+x^3)}} \right| \leq \left| \frac{b^2}{\sqrt{n(n^3)}} \right| = \frac{b^2}{n^{3/2}}. \]

Since \( \sum \frac{1}{n^{3/2}} \) is convergent, by the Weierstrass M-test, it follows that
the series \( \sum_{n=1}^{\infty} \frac{x^2}{\sqrt{n(n^2+x^3)}} \) is U.C. for \(x \in [0,6]\).

Now, suppose \(x\) is large enough. We have:

\[
\frac{x^2}{\sqrt{n(n^2+x^3)}} = \frac{1}{\sqrt{n(n^2+x^3)^{1/3}}} \cdot \frac{1}{\sqrt{n(n^2+x^3)^{1/3}}} \cdot \frac{x^2}{(n^2+x^3)^{1/3}} \leq \frac{1}{\sqrt{n(n^3+x^3)}} \cdot \frac{x^2}{(n+x^3)^{1/3}} = \frac{1}{\sqrt{n+x^3}} \cdot \frac{x^2}{(n+x^3)^{1/3}} = \frac{1}{\sqrt{n}+\frac{x^3}{3}} = \frac{1}{\sqrt{n}+\frac{3}{78}}.
\]

Again, by the Weierstrass M-test, it follows that the
series converge.
Method 2.

We use the following inequality:

**Lemma.** (generalized AM-GM inequality)

Let $0 \leq \delta \leq 1$, and $a, b > 0$. Then we have the following inequality:

$$a^\delta b^{1-\delta} \leq \delta \cdot a + (1-\delta) \cdot b$$

**Proof.** This can be rewritten as: $(\frac{a}{ \delta})^ \delta \leq \delta \cdot (\frac{b}{1-\delta}) + 1 - \delta$,

which follows by Bernoulli's inequality.

We apply this inequality, and we have:

$$\frac{n^2 + x^3}{n^2 + \frac{2}{3} x^3} \geq (n^2)^{\frac{2}{3}} \cdot (x^3)^{\frac{2}{3}}$$

so

$$\frac{n^2 + x^3}{n^2 + \frac{2}{3} x^3} \leq \frac{1}{n^2 + \frac{2}{3} x^3}, \quad x \in (0, \infty).$$

So

$$\frac{x^2}{\ln(n^2 + x^3)} \leq \frac{x^2}{\ln(n^2 + \frac{2}{3} x^3)} = \frac{1}{n^2} \rightarrow$$

the series is uniformly convergent by the Weierstrass $M$-test.

Now, let's prove that the series $\sum_{n=1}^{\infty} \frac{x^3}{\sqrt[n]{n^2 + x^3}}$ is not U.C.

Suppose by contradiction that the series is U.C. on $(0, \infty)$. This means that there exists $\epsilon > 0$ such that

$$\left| \sum_{n=m}^{m+n} \frac{x^3}{\sqrt[n]{n^2 + x^3}} \right| < \epsilon,$$

whenever $m^2 \geq n \geq k \epsilon$ and $x > 0$.

What happens when $x \to \infty$? That is the problem.

We have:

$$\lim_{x \to \infty} \sum_{n=m}^{m+n} \frac{x^3}{\sqrt[n]{n^2 + x^3}} = \lim_{x \to \infty} \frac{1}{\sqrt[2]{1 + \frac{x^3}{n}}} \leq \epsilon,$$

whenever $m^2 \geq n \geq k \epsilon$. 
