SOME USEFUL AND IMPORTANT INTEGRAL INEQUALITIES

Let us consider \( C([a,b]) \) = the set of all continuous functions real-valued defined on the interval \([a,b]\).

Now, let \( h \in C([a,b]) \). Obviously, the following inequality is valid:

\[
\int_a^b h^2(x) \, dx \geq 0, \quad \forall x \in [a,b].
\]

Now, for \( \lambda \in \mathbb{R} \), let us consider \( h(x) = f(x) - \lambda \cdot g(x) \), where \( f, g \in C([a,b]) \). This implies that

\[
\int_a^b (f(x) - \lambda \cdot g(x))^2 \, dx \geq 0.
\]

This last inequality is equivalent with

\[
\int_a^b (f^2(x) - 2 \lambda \cdot f(x) g(x) + \lambda^2 g^2(x)) \, dx \geq 0.
\]

Or

\[
\int_a^b f^2(x) \, dx - 2 \lambda \int_a^b f(x) g(x) \, dx + \lambda^2 \int_a^b g^2(x) \, dx \geq 0.
\]

Define the quadratic function

\[
P(\lambda) = \int_a^b g^2(x) \, dx \cdot \lambda^2 - 2 \lambda \int_a^b f(x) g(x) \, dx + \int_a^b f^2(x) \, dx \geq 0.
\]

Since \( P(\lambda) \geq 0 \) for all \( \lambda \) real, it follows that the discriminant of \( P \) must be negative, i.e.
\[ \Delta = 4 \left( \int_a^b x^2 \, dx \right)^2 - 4 \cdot \int_a^b x \, dx \cdot \int_a^b x^2 \, dx \leq 0 \]

or equivalently

\[ \left( \int_a^b x \, dx \right)^2 \leq \int_a^b x \, dx \cdot \int_a^b x^2 \, dx. \]

In other words, we prove the following inequality:

\[ \text{(C-BS): } \left| \int_a^b f(x)g(x) \, dx \right| \leq \left( \int_a^b f(x) \, dx \right) \cdot \left( \int_a^b g(x) \, dx \right) \]

Where \( f, g \in C([a,b]). \)

Since we mentioned this infamous inequality, I think it's worth giving another proof. This goes as follows:

The main idea is hidden in the symmetrizing substitutions

\[ u \mapsto f(x)g(y) \text{ and } v \mapsto f(y)g(x). \]

Now, from the elementary inequality \( 2uv \leq u^2 + v^2, \)

it follows that

\[ 2 \int_a^b f(x)g(y) \, dx \cdot \int_a^b f(y)g(x) \, dy \leq \int_a^b f(x) \, dx \cdot \int_a^b f(y)g(x) \, dy + \int_a^b g(x) \, dx \cdot \int_a^b f(x)g(y) \, dy. \]

So, integration over \([a,b] \times [a,b]\) yields

\[ 2 \int_a^b \int_a^b f(x)g(y) \, dx \, dy \leq \int_a^b f(x) \, dx \cdot \int_a^b f(x)g(y) \, dy + \int_a^b g(x) \, dx \cdot \int_a^b f(x)g(y) \, dy + \]

\[ \int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx. \]
\[ + \int_a^b f(x)^2 \, dx \cdot \int_a^b g(x)^2 \, dx, \]

which is nothing else than (C-B-S) inequality once it is rewritten with only a single dummy variable.

A natural generalization of the Cauchy-Schwarz's inequality was given by Rogers (1888) and Hölder (1889). Before we state and prove this inequality, we first prove the following

**Lemma.** (Young's inequality)

Let \( a, b > 0 \) and \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, we have the inequality

\[ \frac{a^p}{p} + \frac{b^q}{q} \geq ab. \]

**First proof.**

Since \( \frac{1}{p} + \frac{1}{q} = 1 \), we can take \( p = \frac{m+n}{n} \) and \( q = \frac{m+n}{n} \), where \( m, n \) are positive integers. Now, let \( a = x^\frac{1}{p} \) and \( b = y^\frac{1}{q} \). Then,

\[ \frac{a^p}{p} + \frac{b^q}{q} = \frac{x^{\frac{m+n}{n}}}{\frac{m+n}{n}} + \frac{y^{\frac{m+n}{n}}}{\frac{m+n}{n}} = \frac{m \cdot x + n \cdot y}{m+n}. \]

On the other hand, by weighted arithmetic-geometric mean inequality, we get...
\[
\frac{mx + ny}{m+n} \geq (x^m y^n)^{\frac{1}{m+n}} = x^\frac{m}{m+n}, y^\frac{n}{m+n} = ab,
\]
and thus
\[
\frac{a^p}{p} + \frac{b^q}{q} \geq ab.
\]

**Second solution.**

We prove the following inequality:
\[
(\star) \quad x^t y^{1-t} \leq t \cdot x + (1-t) \cdot y, \quad 0 \leq t \leq 1, \quad x, y > 0.
\]

Equality holds true in (\star) if and only if \( x = y \).

Indeed, the above inequality can be rewritten as
\[
(\star \star) \quad \left(\frac{x}{y}\right)^t \leq 1 - t + t \cdot \frac{x}{y}.
\]

By Bernoulli's inequality, we have
\[
\left(\frac{x}{y}\right)^t = \left[1 + \left(\frac{x}{y} - 1\right)\right]^t \leq 1 + t \left(\frac{x}{y} - 1\right) = 1 - t + t \cdot \frac{x}{y},
\]
and thus (\star) follows immediately.

Now, take \( x = a^p, y = b^q \) and \( t = \frac{1}{p} \in (0,1) \) and by applying (\star \star), we obtain
\[
\left(\frac{a^p}{b^q}\right)^t \leq 1 - \frac{1}{p} + \frac{a^p}{b^q} \cdot \frac{1}{p},
\]
which is equivalent to
\[
\frac{a}{b^{\frac{q}{p}}} \leq 1 - \frac{1}{p} + \frac{1}{p} \cdot \frac{a^p}{b^q},
\]
which gives us
\[
\frac{a}{b^{\frac{q}{p}}} \leq \frac{1}{q} + \frac{1}{p} \cdot \frac{a^p}{b^q}.
\]
and by multiplying with $b^2$, we get
\[ a \cdot b^{2-\frac{1}{p}} \leq \frac{b^2}{q} + \frac{a^p}{p}, \]
so
\[ a \cdot b^{2(1-\frac{1}{p})} \leq \frac{b^2}{q} + \frac{a^p}{p} \quad \text{or} \quad \frac{a^p}{p} + \frac{b^2}{q} \geq ab. \]

Third proof.
We know that the function $t \mapsto \exp(t)$ is convex. This implies that
\[ ab = \exp(\log(ab)) = \exp(\log(a) + \log(b)) = \exp\left(\frac{1}{p} \cdot \log(a^p) + \frac{1}{q} \cdot \log(b^q)\right) \leq \frac{1}{p} \cdot \exp\left(\log(a^p)\right) + \frac{1}{q} \cdot \exp\left(\log(b^q)\right) \]
\[ = \frac{1}{p} \cdot a^p + \frac{1}{q} \cdot b^q, \]
where the last inequality follows from the concavity of the function $t \mapsto \exp(t)$.

Therefore, the lemma is proven. \(\square\)

In this moment, we are in the position to show Hölder's inequality, namely

\[ (\text{Hölder}) : \quad \int_a^b (f(x)g(x))dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \cdot \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}, \]
where $f, g \in C([a, b])$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.\
To simplify the notations, denote $\|f\|_{L^p} = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$.

We are asked to prove that

$$\|fg\|_1 \leq \|f\|_{L^p} \|g\|_{L^{p'}},$$

Put $a = \frac{|f(x)|}{\|f\|_{L^p}}$ and $b = \frac{|g(x)|}{\|g\|_{L^{p'}}}$ in Young's inequality.

Thus, we have

$$\frac{1}{p} \left( \frac{|f(x)|}{\|f\|_{L^p}} \right)^p + \frac{1}{p'} \left( \frac{|g(x)|}{\|g\|_{L^{p'}}} \right)^{p'} \geq \frac{|f(x)| \cdot |g(x)|}{\|f\|_{L^p} \cdot \|g\|_{L^{p'}}},$$

which is equivalent to

$$\frac{1}{p} \cdot \frac{|f(x)|^p}{\|f\|_{L^p}^p} + \frac{1}{p'} \cdot \frac{|g(x)|^{p'}}{\|g\|_{L^{p'}}^{p'}} \geq \frac{|f(x)| \cdot |g(x)|}{\|f\|_{L^p} \cdot \|g\|_{L^{p'}}},$$

and by further computations, we have:

$$\frac{1}{p} \cdot \frac{|f(x)|^p}{\|f\|_{L^p}^p} + \frac{1}{p'} \cdot \frac{|g(x)|^{p'}}{\|g\|_{L^{p'}}^{p'}} \cdot \frac{\|f\|_{L^p} \cdot \|g\|_{L^{p'}}}{\|f\|_{L^p} \cdot \|g\|_{L^{p'}}} \geq \frac{|f(x)| \cdot |g(x)|}{\|f\|_{L^p} \cdot \|g\|_{L^{p'}}},$$

Integrating, we finally obtain

$$\frac{1}{p} \cdot \left( \int_a^b |f(x)|^p \, dx \right) \cdot \|g\|_{L^{p'}} + \frac{1}{p'} \cdot \left( \int_a^b |g(x)|^{p'} \, dx \right) \cdot \|f\|_{L^p} \geq \int_a^b |f(x)g(x)| \, dx,$$

This gives us
\[
\frac{1}{p} \cdot \frac{\|f\|_p^p}{\|f\|_p^{p-1}} + \frac{1}{q} \cdot \frac{\|g\|_q^q}{\|g\|_q^{q-1}} \geq \|fg\|_1,
\]
or
\[
\|fg\|_1 \cdot \|g\|_q \left( \frac{1}{p} + \frac{1}{q} \right) \geq \|fg\|_1, \text{ and we are done.}
\]

Remarks.

1. Let us take \( q = \frac{p}{p-1} \) (Hölder exponent). Hölder’s inequality can be rewritten as
\[
\int_a^b f(x)g(x)dx \leq \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int_a^b |g(x)|^q dx \right)^{\frac{1}{q}}.
\]

Also, for \( p = 2 \), we obtain Cauchy-Schwarz’s inequality!

2. Suppose that \( p, q \in (0, +\infty) \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \). Then, we have the following extension of Hölder’s inequality,
\[
\|fg\|_r \leq \|fg\|_1 \cdot \|g\|_q
\]
or equivalently
\[
\left( \int_a^b |f(x)g(x)|^r dx \right)^{\frac{1}{r}} \leq \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int_a^b |g(x)|^q dx \right)^{\frac{1}{q}}.
\]

This inequality follows immediately from the usual Hölder inequality if we apply it for the functions...
Indeed, we apply Hölder's inequality with exponents, $p_1 = \frac{\ell}{2}$ and $q_1 = \frac{2}{\ell}$, where $\frac{1}{p_1} + \frac{1}{q_1} = 1$, we have

\[
\left( \int_a^b \left| f(x)^{p_1} g(x)^{q_1} \right|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_a^b \left| f(x)^{p_1} \right|^2 \, dx \right)^{\frac{1}{2}} \cdot \left( \int_a^b \left| g(x)^{q_1} \right|^2 \, dx \right)^{\frac{1}{2}}
\]

which gives us exactly our extension of Hölder's inequality!

3. In fact, one can prove the following generalized version of Hölder's inequality:

\[
\text{If } \sum_{k=1}^\infty \frac{1}{p_k} = \frac{1}{2}, \text{ with } p_k, n \geq 1, \text{ then }\\
\|f_1 f_2 \cdots f_n\|_{\ell^2} \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_n\|_{p_n}.
\]

The case $k=2$ was proved earlier! We proceed by induction. Assume that the inequality is true for some $N > 0$. Now, applying the case $k=2$ for the functions $f = f_1 f_2 \cdots f_N$ and $g = f_{N+1}$, we have:

\[
\|f_1 f_2 \cdots f_N f_{N+1}\|_{\ell^2} \leq \|f_1 f_2 \cdots f_N\|_{p_1} \|f_{N+1}\|_{p_{N+1}}.
\]

Now, by the induction hypothesis, taking $p = \left( \sum_{k=1}^N \frac{1}{p_k} \right)^{-1}$, we have the inequality for the norm $\| f_1 f_2 \cdots f_N f_{N+1} \|_{p_{N+1}}$ and the generalization follows immediately. $\square$
In what follows, we present some applications of H"older's inequality. We start with

\[
\text{(Minkowski first inequality): (triangle inequality)}
\]

\[
\left( \int_a^b |f(x)| + |g(x)|^p \, dx \right)^{\frac{1}{p}} \leq \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_a^b |g(x)|^p \, dx \right)^{\frac{1}{p}}.
\]

Equivalently, this can be rewritten as

\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

Indeed, we have

\[
\|f + g\|_p^p = \left( \int_a^b |f(x)| + |g(x)|^p \, dx \right) = \left( \int_a^b |f(x)| + |g(x)|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_a^b |g(x)|^p \, dx \right)^{\frac{1}{p}} = \int_a^b |f(x)|^p \, dx + \int_a^b |g(x)|^p \, dx,
\]

where we applied triangle inequality.

Now, by H"older's inequality, we obtain

\[
\int_a^b |f(x)| \cdot |f(x)|^{\frac{1}{p}} \, dx \leq \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \cdot \left( \int_a^b |f(x)|^{\frac{1}{p}} \, dx \right)^{1-\frac{1}{p}}
\]

\[
= \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \cdot \left( \int_a^b |f(x)|^{\frac{1}{p}} \, dx \right)^{1-\frac{1}{p}}.
\]

Similarly, we obtain
\[ \int_{a}^{b} |f(x) + g(x)|^{p+1} \, dx \leq \left( \int_{a}^{b} |f(x)|^{p} \, dx \right)^{1 - \frac{1}{p}} \left( \int_{a}^{b} |g(x)|^{p} \, dx \right)^{\frac{1}{p}} \left( \int_{a}^{b} |f(x) + g(x)|^{p} \, dx \right)^{\frac{1}{p}}. \]

Therefore, we have

\[ \|f + g\|_{p}^{p} \leq \left( \int_{a}^{b} |f(x) + g(x)|^{p} \, dx \right)^{1 - \frac{1}{p}} \left( \int_{a}^{b} |f(x)|^{p} \, dx \right)^{\frac{1}{p}} + \left( \int_{a}^{b} |g(x)|^{p} \, dx \right)^{\frac{1}{p}}, \]

or equivalently

\[ \|f + g\|_{p}^{p} \leq \|f\|_{p}^{p} \|f + g\|_{p} \] (\|f\|_{p} + \|g\|_{p}), \]

which finally gives us

\[ \|f + g\|_{p} \leq \|f\|_{p} + \|g\|_{p} \]

whenever \( \|f + g\|_{p} \) is nonzero and finite.

If \( \|f + g\|_{p} = 0 \), then there is nothing to prove. Also, let's note that \( \|f + g\|_{p} \) is always finite since by the convexity of \( x \mapsto x^{p}, p > 1 \), we have the inequality

\[ \|f + g\|_{p} \leq 2^{p-1} (\|f\|_{p} + \|g\|_{p}). \]

\( \blacksquare \)

(Young inequality), or (generalized Minkowski inequality)

\[ \left( \int_{a}^{b} \left| \int_{c}^{d} f(x, y) \, dy \right|^{p} \, dx \right)^{\frac{1}{p}} \leq \int_{c}^{d} \left( \int_{a}^{b} \left| f(x, y) \right|^{p} \, dx \right)^{\frac{1}{p}} \, dy. \]

\( 1 \leq p < \infty. \)
Proof. The case $p=1$ follows from Fubini's theorem, so let us assume that $p>1$ and note that
\[
\int_a^b \int_c^d f(x,y) \, dy \, dx = \int_a^b \int_c^d f(x,y) \, dy \, dx \leq \int_a^b \int_c^d f(x,y) \, dy \, dx \leq \int_a^b \int_c^d \int_0^1 |f(x,y)| \, dy \, dx = \\
\int_a^b \left( \int_c^d \int_0^1 |f(x,y)| \, dy \right) \, dx = \\
\int_a^b \left( \int_c^d \int_0^1 \frac{d}{dx} f(x,y) \right) \, dx = \\
\int_a^b \left( \int_c^d \int_0^1 \frac{d}{dx} f(x,y) \right) \, dx.
\]
Fubini's theorem!

Our intention is to apply Hölder's inequality to the inner integral (with respect to $x$), indeed, we have:
\[
\int_a^b \int_c^d f(x,t) \, dt \, dx \leq \left( \int_a^b \int_c^d |f(x,t)|^p \, dt \, dx \right)^{\frac{1}{p}} \cdot \left( \int_a^b \int_c^d 1 \, dx \right)^{\frac{1}{p-1}}.
\]
Hölder's inequality

\[
\cdot \left( \int_a^b \int_c^d 1 \, dx \right)^{\frac{1}{p-1}} = \\
\left( \int_a^b \int_c^d |f(x,t)|^p \, dt \, dx \right)^{\frac{1}{p}} \cdot \left( \int_a^b \int_c^d 1 \, dx \right)^{\frac{1}{p-1}}.
\]

Now, using the two inequalities we obtained, we derive...
\[ \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx \leq \int_a^b \left( \int_c^d f(x,t) \, dt \right) \left( \int_a^b f(x,y) \, dx \right) \, dy \]

\[ = \left( \int_a^b \int_c^d f(x,t) \, dt \right)^{\frac{1}{p}} \cdot \left( \int_a^b \left( \int_c^d f(x,y) \, dx \right)^p \, dy \right)^{\frac{1}{q}} \]

Now, if we divide by the number \( \left( \int_a^b \int_c^d f(x,t) \, dt \right)^{\frac{1}{p}} \) and taking into account that \( 1 - \frac{1}{2} = \frac{1}{2} \), we obtain our inequality. \( \square \)

**Question:**

Does the generalized Minkowski imply the Minkowski inequality?

**Answer:** YES!

Let us recall the classical Minkowski inequality:

\[ \| f_1 + f_2 \|_p \leq \| f_1 \|_p + \| f_2 \|_p. \]

Indeed, we have:

\[ \| f_1 + f_2 \|_p = \left( \int_a^b \int_c^d | f_1(x,y) + f_2(x,y) |^p \, dy \, dx \right)^{\frac{1}{p}} \leq \left( \int_a^b \int_c^d \left( | f_1(x,y) |^p + | f_2(x,y) |^p \right) \, dy \, dx \right)^{\frac{1}{p}} \]

\[ = \| f_1 \|_p + \| f_2 \|_p. \]

Here \( f_i(y) = f(x,y) \) for \( i = 1,2 \).