

SOME USEFUL AND IMPORTANT INTEGRAL INEQUALITIES

Let us consider $C([a, b]) =$ the set of all continuous functions real-valued defined on the interval $[a, b]$.

Now, let $h \in C([a, b])$. Obviously, the following inequality is valid:

$$\int_a^b h^2(x) dx \geq 0, \forall x \in [a, b].$$

Now, for $\lambda \in \mathbb{R}$, let us consider $h(x) = f(x) - \lambda \cdot g(x)$, where $f, g \in C([a, b])$. This implies that

$$\int_a^b (f(x) - \lambda \cdot g(x))^2 dx \geq 0.$$

This last inequality is equivalent with

$$\int_a^b (f^2(x) - 2\lambda \cdot f(x)g(x) + \lambda^2 g^2(x)) dx \geq 0$$

$$\text{or} \quad \int_a^b f^2(x) dx - 2\lambda \int_a^b f(x)g(x) dx + \lambda^2 \int_a^b g^2(x) dx \geq 0.$$

Define the quadratic function

$$P(\lambda) = \int_a^b g^2(x) dx \cdot \lambda^2 - 2 \int_a^b f(x)g(x) dx \cdot \lambda + \int_a^b f^2(x) dx.$$

Since $P(\lambda) \geq 0$ for all λ real, it follows that the discriminant of P must be negative, i.e.