Problem. (University of Pittsburgh Prelim, 2002)
Prove that if $f: \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, i.e.,
$$|f(x) - f(y)| \leq L \cdot |x - y|$$
for all $x, y \in \mathbb{R}^n$ and some $L > 0$ and $E \subset \mathbb{R}^n$ is a set of measure zero, then $f(E) \subset \mathbb{R}^m$ is a set of measure zero. (This is called Luzin N property)

Solution.
Consider the ball $B(x, r)$ and let $y \in B(x, r)$. Since $f$ is a Lipschitz function, we have:
$$|f(y) - f(x)| \leq L \cdot |y - x| \leq L \cdot r$$
This means that $f(y) \in B(f(x), Lr)$, so $f(B(x, r)) \subseteq B(f(x), Lr)$.

Now, for $\varepsilon > 0$, fixed and since $\text{meas}(E) = 0$, there exists a collection of balls $B(x_k, r_k)$ such that $E \subseteq \bigcup B(x_k, r_k)$ and
$$\sum_{k=1}^{\infty} r_k < \frac{\varepsilon}{L}.$$ It follows that
$$f(E) \subseteq f\left(\bigcup_{k=1}^{\infty} B(x_k, r_k)\right) = \bigcup_{k=1}^{\infty} f(B(x_k, r_k)) \subseteq \bigcup_{k=1}^{\infty} B(f(x_k), Lr_k).$$
Moreover, $\sum_{k=1}^{\infty} (Lr_k)^n = L^n \cdot \sum_{k=1}^{\infty} r_k^n < \varepsilon$ and we are done. \(\Box\)
problem.
Recall \( f : X \to Y \) is \( \alpha \)-Hölder continuous is
\[
\delta(f(x), f(y)) \leq C \cdot d(x, y)^\alpha,
\]
for all \( x, y \in X \), where \( \delta \) and \( d \) are metrics on
\( X \) and \( Y \) respectively. Hence that if \( g : [0,1] \to [0,1]^2 \)
is an \( \alpha \)-Hölder continuous mapping whose image
is the entire square \([0,1]^2\), then \( \alpha \leq \frac{1}{2} \).

Solution.
Assume by contradiction that \( \alpha > \frac{1}{2} \). Consider
an arbitrary \( \varepsilon > 0 \) and \( N = \lceil \frac{1}{2\varepsilon} \rceil \). Since \( g \) is Hölder
continuous we have that for all \( x, y \in [0,1] \) with
\( |x - y| < \varepsilon \), \( \| g(x) - g(y) \| \leq C \cdot \varepsilon^\alpha \).

Now the main idea is to cover \( g([0,1]) \) with
\( N \) balls of radius \( C \cdot \varepsilon^{\frac{1}{2} - \alpha} \) and centered in \( R \cdot \delta \),
\( 0 \leq k \leq N \).

We obtain:
\[
\text{meas}(g([0,1])) \leq N \cdot (C^2 \cdot \varepsilon^{2\alpha}) = N \cdot C \cdot \varepsilon^\alpha =
\]
\[
= N \cdot C \cdot \varepsilon^{\frac{1}{2} - \alpha} \leq C \cdot \varepsilon^{\frac{1}{2} - \alpha + \frac{1}{2}}.
\]

By our assumption.

Now, for \( \varepsilon > 0 \) fixed, and for \( \delta = \left( \frac{\varepsilon}{C} \right)^{\frac{1}{2\alpha - 1}} \), we have
\[
\text{meas}(g([0,1])) \leq \varepsilon \Rightarrow \text{meas}(g([0,1])) = 0.
\]

On the other hand:
\[
\text{meas}(g([0,1])) = \text{meas}(\{ [0,1] \times [0,1] \}) = 1.
\]
Another solution.

Lemma.
Suppose that $f$ is defined on a compact set $E \subset \mathbb{R}^n$ is $\beta$-Hölder continuous. Then, we have:

\[ \mathcal{H}_\beta(f(E)) \leq M \beta \mathcal{H}_\alpha(E), \text{ if } \beta < \frac{\alpha}{n} \]

\[ \dim f(E) \leq \frac{1}{\beta} \dim (E). \]

Proof of the lemma.
Suppose $\{E_k\}$ is a countable family of sets that covers $E$ and $\mathcal{H}_\alpha(E)$ is finite. Then $f(\bigcup_k E_k)$ covers $f(E)$ and moreover $f(\bigcup_k E_k)$ has diameter less than $M \cdot (\text{diam } E_k)$. Therefore

\[ \sum_k \left( \text{diam } f(\bigcup_k E_k) \right)^{-\frac{\alpha}{n}} \leq M^{-\frac{\alpha}{n}} \sum_k \left( \text{diam } E_k \right)^{-\frac{\alpha}{n}}, \]

and therefore our first assertion of the lemma follows. The second part follows immediately from the first part.

Now, going back to our problem, everything will be a consequence of the second part of the lemma. Indeed, since our function $f$ is $\alpha$-Hölder continuous, we have

\[ 2 - \dim f([0,1]) \leq \frac{1}{\alpha} \dim [0,1] = \frac{1}{2}, \]

and with the conclusion. $\square$
Problem. (American Mathematical Monthly, 2010)

Prove that there is no function $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with the property

$$\|f(x) - f(y)\| \geq \|x - y\|,$$

for $x, y \in \mathbb{R}^3$.

Solution.

Let $x \in \mathbb{R}^3$ such that $f(x) = f(y)$. It follows that $f$ is injective because

$$\|x - y\| \leq \|f(x) - f(y)\| = 0 \Rightarrow x = y.$$

Therefore, the inverse of $f$ exists. Moreover, $f^{-1}: \text{Im}(f) \rightarrow \mathbb{R}^3$ is surjective. Now, let's define $g: \text{Im}(f) \times \mathbb{R} \rightarrow \mathbb{R}^3$, by

$$g(x, y, z) = f^{-1}(x, y).$$

It is clear that $g$ is a Lipschitz function because

$$\|g(x, y, z) - g(m, n, p)\| = \|f^{-1}(x, y) - f^{-1}(m, n)\|$$

$$\leq \|g(x, y, z) - (m, n)\|$$

$$\leq \|g(x, y, z) - (m, n, p)\|$$

$f^{-1}$ is Lipschitz.

Now, the measure of $\text{Im}(f) \times f^{-1}$ is zero. On the other hand, $g(\text{Im}(f) \times f^{-1})$ has measure zero, which is in contradiction with the fact that $g(\text{Im}(f) \times f^{-1}) = \mathbb{R}^3$ which has measure $\infty$. $\blacksquare$
Problem. (Pitt Helim, 2012)

Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a continuous function. Let \( K \subset \mathbb{R}^3 \) be a compact set such that
\[
|f(x) - f(y)| \leq 2012 |x - y|^2, \quad \forall x, y \in K.
\]
Prove that the set \( f(K) \subset \mathbb{R}^3 \) has measure zero.

Solution. A subset of \( \mathbb{R}^2 \).

Since \( K \) is compact, \( K \subset [-M, M]^3 \) for some \( M > 0 \).
Hence \( K \) can be covered by \( n^3 \) closed cubes of side-length \( \frac{2M}{n} \) and thus of diameter \( \frac{2\sqrt{3}M}{n} \).

Each such cube can be covered by \( k(n) \leq n^3 \) closed balls of \( B(x_{in}, \frac{2\sqrt{3}M}{n}) \) centered at \( x_{in} \in K \),
where \( C = 2\sqrt{3}M \), does not depend on \( n \). Indeed, we only cover these cubes from the partition of \( [-M, M]^3 \) that have nonempty intersection with \( K \) and \( k(n) \leq n^3 \) is the number of such cubes.

It follows from the assumptions that
\[
\mu \left( K \cap B(x_{in}, \frac{2\sqrt{3}M}{n}) \right) < \frac{1}{2012} \left( \frac{2\sqrt{3}M}{n} \right)^2 n^2 \leq \frac{1}{2012} c^2 n^2.
\]

and hence \( f(K) \) can be covered by \( k(n) \) closed balls of radius \( \frac{c}{n} \), where \( c = \sqrt{2} \) does not depend on \( n \), since the sum of squares of the radii of balls covering \( f(K) \) satisfies
\[
k(n) \cdot (\frac{c}{n})^2 n^2 \leq (\frac{c}{2})^2 n^2 \to 0, \quad n \to \infty.
\]

\( \Rightarrow |f(K)| = 0. \quad \square \)
Problem. (University of Pittsburgh Prelim Exam, 2008)

Assume that \( f: [0,1] \rightarrow \mathbb{R} \) is a continuous function such that the set \( \{ x \in [0,1] : f(x) = 1 \} \) has measure zero. Prove directly (without using results like monotone or dominated convergence theorems) that

\[
\lim_{n \to \infty} \int_0^1 (f(x))^n \, dx = 0.
\]

Solution.

Let \( N = \{ x \in [0,1] : f(x) = 1 \} \). Since \( N \) has measure 0, it follows that there exist \( (a_i, b_i) \) such that

\[
\bigcup_{i=1}^{\infty} (a_i, b_i) \supseteq N \quad \text{and} \quad \sum_{i=1}^{\infty} (b_i - a_i) < \frac{\varepsilon}{M}.
\]

Let \( A = \bigcup_{i=1}^{\infty} (a_i, b_i) \). Now, \( A \) is open, hence \( [0,1] \setminus A \) is closed and bounded since it's compact. Therefore, \( f \) attains a max on \( [0,1] \setminus A \), let's say \( M \). Then \( M < 1 \), we've excluded \( A \).

Note also that \( |f(x)| < M \), so \( |f(x)^n| = |f(x)|^n < M^n \) on \( [0,1] \setminus A \). Now, take \( N \) such that for all \( x > N \), \( M^n < \frac{\varepsilon}{2} \). Consider the characteristic function

\[
\chi_A = \begin{cases} 
1, & x \in A \\
0, & x \in A^c = [0,1] \setminus A
\end{cases}
\]

\[
\chi_{A^c} = \begin{cases} 
0, & x \in A \\
1, & x \in A^c
\end{cases}
\]
Clearly $x_a$ and $x_{Ac}$ are both integrable on $[0,1]$ since they are discontinuous at $a_i, b_i$, of which there are a countable number. Thus

$$\int_0^1 f(x) \, dx = \int_0^1 x_a + x_{Ac} \, dx = \int_0^1 f(x) \, dx$$

$$\leq \int_0^1 x_a \, dx + \int_0^1 x_{Ac} \, dx \leq \int_0^1 x_a \, dx + \int_0^1 x_{Ac} \, dx \leq \int_0^1 x_a \, dx + \frac{1}{n} \int_0^1 f(x) \, dx$$

$$= \int_0^1 x_a \, dx + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Problem.

Let \( Q = \{0, 1\}^n \) and \( f : Q \to \mathbb{R} \) be continuous on \( Q \). For each \( k \in \mathbb{N} \), let \( f^k \) denote the \( k \)-th power of \( f \). Suppose that

\[
\sum_{x \in Q} f^k = 1
\]

holds for \( k = 2, 3 \) and 4. Prove that \( f(x) = \frac{1}{n} \) for every \( x \in Q \).

Solution.

We have

\[
\sum_{x \in Q} (f^2 - 1)^2 = 0 \implies (f^2 - 1)^2 = 0.
\]

Thus \( f : Q \to \{-1, 1\} \). Since \( Q \) is convex, it is connected, and therefore the continuous function \( f \) must be constant, so \( f = -1 \) or \( f = 1 \). From

\[
\sum_{x \in Q} f^3 = +1 \implies f = 1. \quad \square
\]