Math 0240
Summer 2017
Second Exam
6/22/2017
Time Limit: 2 Hours + $\epsilon$, $\epsilon \leq 1$ hour
Class: Calculus III

This exam contains 12 pages (including this cover page) and 11 questions. Total of points is 33. The last problem is for BONUS points.
This is NOT an open book and notes exam. No calculators are allowed. Show all your work (no work = no credit). Write neatly. Simplify your answers.

Grade Table (for teacher use only)

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Total</td>
<td>33</td>
<td>33</td>
</tr>
</tbody>
</table>

Good luck !!!
1. (3 points) Find the maximum and the minimum of \( f(x, y) = 5x - 3y \) subject to the constraint \( x^2 + y^2 = 136 \).

It is clear from the constraint that the region of possible solutions lies on a disk of radius \( \sqrt{136} \), which is a closed and bounded region and hence by the Extreme Value Theorem, we know that minimum and maximum values exist. Define the Lagrangian,

\[
L(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y) = (5x - 3y) - \lambda(x^2 + y^2 - 136).
\]

We have:

\[
\begin{align*}
\frac{\partial L}{\partial x} &= 5 - 2\lambda x = 0 \\
\frac{\partial L}{\partial y} &= -3 - 2\lambda y = 0 \\
\frac{\partial L}{\partial \lambda} &= x^2 + y^2 - 136 = 0
\end{align*}
\]

It is evident that \( \lambda \neq 0 \). This implies that \( x = \frac{5}{2\lambda}, y = -\frac{3}{2\lambda} \).

Plugging these values into the constraint gives, \( \frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = 136 \) which gives \( \lambda = \pm \frac{3}{4} \).

Solving for \( \lambda \), we have \( \lambda^2 = \frac{1}{16} \Rightarrow \lambda = \pm \frac{1}{4} \).

If \( \lambda = -\frac{1}{4} \), we get \( (x, y) = (-10, 6) \) and if \( \lambda = \frac{1}{4} \), we get \( (x, y) = (10, -6) \). Moreover, we have

\[
f(-10, 6) = -68 \rightarrow \text{minimum at } (-10, 6)
\]

and

\[
f(10, -6) = 68 \rightarrow \text{maximum at } (10, -6).
\]

\( \square \)
2. (3 points) A cardboard box without a lid is to have a volume of 8000 $cm^3$. Find the dimensions that minimize the amount of cardboard used.

Denote $V = 8000$ and let $x$ be the width, $y$ be the length, and $z$ be the height. Then $V = xyz$ and area is given by $A = xy + 2xz + 2yz$. Since $z = \frac{V}{xy}$, we consider the function $A(x, y) = xy + \frac{2V}{x} + \frac{2V}{y}$ and we need to find the minimum!

Note that $x, y > 0$ and $A_x = y - 2\sqrt{\frac{V}{x^2}} = 0, A_y = x - 2\sqrt{\frac{V}{y^2}} = 0$, so this easily implies that $x = y = 3\sqrt{2V}$. The critical point is $(\sqrt{2V}, \sqrt{2V})$. Moreover, $A_{xx} = 4\sqrt{\frac{V}{x^3}}, A_{yy} = 1$. At the critical point $A_{xx} = 2V, A_{yy} = 2V$. Then

$$D = \begin{vmatrix}
A_{xx} & A_{xy} \\
A_{yx} & A_{yy}
\end{vmatrix} = 4V^2 - 1 > 0.$$

Also, $A_{xx} = 2V > 0$. Thus, $A(x, y)$ attains a local minimum at the critical point which is the absolute minimum. We have $3\sqrt{2V} = 3\sqrt{4000} \Rightarrow x = y = 20\sqrt{3} \text{ cm}$ and $z = 10\sqrt{3} \text{ cm}$.

Second solution. One can obtain the minimum of $A(x, y)$ by applying the arithmetic-geometric mean inequality. We have $A(x, y) = xy + \frac{2V}{x} + \frac{2V}{y} \geq 3\sqrt[3]{xy \cdot \frac{2V}{x} \cdot \frac{2V}{y}} = 3(4V)^{\frac{3}{2}}$.

Moreover, the minimum is attained when $x = y = 3\sqrt{2V}$, so $(x, y) = (\sqrt{2V}, \sqrt{2V}) = (20\sqrt{3}, 20\sqrt{3})$ and this gives us $z = 10\sqrt{3} \text{ cm}$.

Third solution. (Using Lagrange multipliers).

We have $V(x, y, z) = xyz$ and $A(x, y, z) = xy + 2yz + 2xz$. By the Lagrange multipliers method, we have $\nabla V = \lambda \nabla A$, and this gives us...
2. (3 points) A cardboard box without a lid is to have a volume of 8000 cm³. Find the dimensions that minimize the amount of cardboard used.

We have to minimize the function \( A(x,y,z) = xy + 2yz + 2xz \) subject to the constraint \( V(x,y,z) = xyz = 8000 \).

Consider the Lagrangian

\[
L(x,y,z,\lambda) = A(x,y,z) - \lambda V(x,y,z) = (xy + 2yz + 2xz) - \lambda (xyz - 8000)
\]

\[
\begin{align*}
L_x &= y + 2z - \lambda yz = 0 \\
L_y &= x + 2z - \lambda xz = 0 \\
L_z &= 2y + 2x - \lambda xy = 0 \\
L_\lambda &= xyz - 8000 = 0
\end{align*}
\]

\[
\begin{align*}
y + 2z &= \lambda yz \\
x + 2z &= \lambda xz \\
2y + 2x &= \lambda xy \\
xyz &= 8000
\end{align*}
\]

\[
\begin{align*}
x + 2z &= \lambda xz \\
y + 2z &= \lambda yz \\
2y + 2x &= \lambda xy \\
\Rightarrow 2x &= 2z \\
\Rightarrow x &= z
\end{align*}
\]

Then, \( 4x = \lambda x^2 \Rightarrow 2x^2 - 4x = 0 \Rightarrow x(2x - 4) = 0 \Rightarrow 2x = 4. \)

This implies that \( x = 2z. \) Therefore, we have \( x = y = z \), which will give us \( (x,y,z) = (20 \sqrt{2}, 20 \sqrt{2}, 10 \sqrt{2}) \) which are the dimensions that minimize the amount of cardboard used. \( \square \)
3. (3 points) John happens to acquire 420 feet of fencing and decides to use it to start a
kennel by building 5 identical adjacent rectangular runs. Find the dimensions of each
run that maximizes the area.

\[ \text{we let the area of a run, and let} \]
\[ x, y \text{ be the dimensions of each run.} \]

Clearly, there are 10 sections of fence corresponding to widths \( x \) and 6 sections of
fence corresponding to lengths \( y \).

Therefore, we want to maximize \( A = xy \) subject to the constraint
\[ 10x + 6y = 420. \]

Since \( x \) and \( y \) are positive, we only need to find
absolute extrema for \( x \) in \([0, 42]\). The Lagrangian is given

\[ L(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y) = xy - \lambda (10x + 6y - 420). \]

We have
\[ L_x = y - 10\lambda, \quad L_y = x - 6\lambda, \quad \text{and} \quad L_\lambda = 10x + 6y - 420. \]

The critical points of \( L(x, y, \lambda) \) satisfy
\[ y = 10\lambda, \quad x = 6\lambda, \quad 10x + 6y = 420. \]

The first two equations parametrize the extrema in the parameter \( \lambda \), which is why
we eliminate \( \lambda \) to obtain \( \frac{y}{10} = \frac{x}{6} = \frac{5x}{3} \). Substituting into the constraint
we obtain
\[ 10x + 6 \left( \frac{5x}{3} \right) = 420, \]
and thus \( x = 21 \) feet. Also, we have
\[ y = 5 \cdot \frac{21}{3} = 35 \text{ feet.} \]
At the critical point \((xy) = (21, 35)\), the area is 735 square feet.

Another solution.

Again, one can use the arithmetic-geometric mean inequality
\[ \frac{420 - 10x + 6y}{2} \geq \sqrt{10x \cdot 6y} = 2\sqrt{10xy} \quad \text{or} \quad 210 \geq \sqrt{10xy}, \]
which implies \( 60x \cdot y \leq 210^2 \) or \( xy \leq 735 \) square feet. The equality

\[ x = 21 \quad \text{and} \quad y = 35. \quad \square \]
4. (3 points) Compute the integral

\[ \int \int_R (x^2y^2 + \cos(\pi x) + \cos(\pi y)) \, dA \]

where \( R = [-2, -1] \times [0, 1] \).

We have

\[
\int \int_R (x^2y^2 + \cos(\pi x) + \cos(\pi y)) \, dA = \int_{-2}^{1} \int_{0}^{1} (x^2y^2 + \cos(\pi x) + \sin(\pi y)) \, dx \, dy
\]

\[
= \int_{0}^{1} \left[ \frac{1}{3} x^3 y^2 + \frac{1}{\pi} \sin(\pi x) + x \cdot \sin(\pi y) \right]_{-2}^{1} \, dy
\]

\[
= \int_{0}^{1} \left( \frac{1}{3} y^2 + \sin(\pi y) \right) \, dy = \frac{1}{3} y^2 - \frac{1}{\pi} \cos(\pi y) \bigg|_{0}^{1} = \frac{7}{3} + \frac{2}{\pi}.
\]
5. (3 points) Evaluate the integral \( I = \int_0^2 \int_0^{\sqrt{2x-x^2}} y \sqrt{x^2 + y^2} \, dy \, dx \) by converting to polar coordinates.

We have to find the domain of integration. Indeed, let us note that \( 0 \leq x \leq 2 \) and \( 0 \leq y \leq \sqrt{2x-x^2} \). This gives us \( x^2 + y^2 = 2x \). (*) This last equality can be also rewritten as \( (x-1)^2 + y^2 = 1 \) which is a circle of radius 1. centered at \((1,0)\).

In polar coordinates (*) is equivalent with \( r^2 = 2 \cos \theta \) or \( r = 2 \cos \theta \).

Therefore, our domain is given by

\[
D = \{(r, \theta) : 0 \leq \theta \leq \frac{
}{2}, \ 0 \leq r \leq 2 \cos \theta\}
\]

Hence, \( I = \int_0^{\pi/2} \int_0^{2 \cos \theta} r^3 \sin \theta \, dr \, d\theta = \int_0^{\pi/2} \sin \theta \left( \frac{r^4}{4} \right)_{r=0}^{r=2 \cos \theta} \, d\theta = \int_0^{\pi/2} \frac{1}{4} \sin \theta \cdot (2 \cos \theta)^4 \, d\theta = 4 \int_0^{\pi/2} \sin \theta \cdot (\cos \theta)^4 \, d\theta = 4 \left[ -\frac{(\cos \theta)^5}{5} \right]_0^{\pi/2} = \frac{4}{5} \square \)
6. (3 points) Find the volume of the solid $E$ bounded by $y = x^2$, $x = y^2$ and $z = x + y + 5$, and $z = 0$.

We have $E = \{(x,y,z) : 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}, 0 \leq z \leq x + y + 5\}$. The volume $V$ of the solid $E$ is given by

$$V = \iiint_E dV = \int_0^1 \int_{\sqrt{x}}^{\frac{1}{\sqrt{x}}} \int_0^{x+y+5} dz \, dy \, dx = \int_0^1 \int_{\sqrt{x}}^{x+y+5} (x+y+5) \, dy \, dx =$$

$$= \int_0^1 \left( (x+5)y + \frac{y^2}{2} \right)_{\sqrt{x}}^{x+y+5} \, dx = \int_0^1 \left( x^2 + 5x + \frac{1}{2}x - x^3 - 5x^2 - \frac{1}{2}x^4 \right) \, dx$$

$$= \left( \frac{2}{5}x^{\frac{5}{2}} + \frac{10}{3}x^{\frac{3}{2}} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{5}{3}x^3 - \frac{1}{10}x^5 \right) \bigg|_0^1 =$$

$$= \frac{2}{5} + \frac{10}{3} + \frac{1}{4} - \frac{1}{4} - \frac{5}{3} - \frac{1}{10} = \frac{59}{30}.$$
7. (3 points) Evaluate the integral

\[ \iiint_{\Omega} \frac{yz}{\sqrt{x^2 + y^2}} \, dx \, dy \, dz, \]

where \( \Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + \frac{y^2}{4} \leq 1, x^2 + y^2 \geq 1, x \geq 0, y \geq 0, 0 \leq z \leq 5\}. \)

In cylindrical coordinates we have

\[
\begin{cases}
    x = r \cos \theta, \quad r > 0 \\
y = r \sin \theta, \quad \theta \in [0, 2\pi] \\
z = z
\end{cases}
\]

The Jacobian is given by the \( J = r \). Also, our "cylindrical domain is

\( \mathcal{D}_r = \{(r, \theta, z) : 0 \leq z \leq 5, 0 \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq \frac{2}{\sqrt{1 + 3 \cos^2 \theta}} \}. \)

Therefore, our integral becomes

\[
\iiint_{\mathcal{D}_r} \frac{yz}{r} \, dx \, dy \, dz = \int_0^5 dz \int_0^{\frac{\pi}{2}} d\theta \int_1^{\frac{2}{\sqrt{1 + 3 \cos^2 \theta}}} \frac{2 r \sin \theta}{Mr} \, dr
\]

\[
= \frac{25}{2} \int_0^{\frac{\pi}{2}} \frac{3(1 - \cos^2 \theta)}{3 \cos^2 \theta + 1} \, \sin \theta \, d\theta = \frac{25}{2} \int_1^\infty \frac{3u^2 - 4}{3u^2 + 1} \, du
\]

\[
= \frac{25}{2} \left( \int \frac{3u^2 + 4}{3u^2 + 1} \, du - 4 \int \frac{1}{3u^2 + 1} \, du \right) = \frac{25}{18} (4\sqrt{3}\pi - 9).
\]

\( \Box \)
8. (3 points) Find the value of the integral

\[ \int_C -5x^2\,dx + 7xy\,dy \]

where \( C \) is the closed curve consisting of the edges of the triangle with vertices (0,0), (3,1) to (0,3) oriented counterclockwise.

We have \( P(x,y) = -5x^2 \) and \( Q(x,y) = 7xy \). By applying Green's theorem, we obtain

\[ \int_C -5x^2\,dx + 7xy\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \,dA = \iint_D \left( 7y - 0 \right) \,dA = \int_0^3 \int_0^{\frac{2}{3}x+3} 7y \,dy \,dx = \int_0^3 \frac{7}{2} \left( \frac{2}{3}x+3 \right)^2 \,dx = \frac{7}{2} \left( 3 \cdot 0 - 2 \cdot 0 + \frac{4}{9} \cdot 0^2 - \frac{4}{9} \cdot 3^2 \right) = 42. \]
9. (3 points) Evaluate

\[ \int \int_S \mathbf{F} \cdot d\mathbf{S} \]

if \( \mathbf{F}(x, y, z) = (yz) \mathbf{i} + (x^2y) \mathbf{j} + (4zx^2) \mathbf{k} \) and \( S \) is the surface of the solid bounded by the upper hemisphere \( x^2 + y^2 + z^2 = 1, \ z \geq 0 \), and the plane \( z = 0 \) with the normal pointing outward.

We have

\[ \int \int_S F \cdot dS = \iiint_{E} \text{div} \mathbf{F} \, dV = \iiint_{E} 0 + x^2 + 4x^2 \, dV. \]

Using spherical coordinates, we have

\[ x = 3 \sin \phi \cos \theta \]
\[ y = 3 \sin \phi \sin \theta \]
\[ z = 3 \cos \phi \]

We have that

\[ \iiint_{E} 5x^2 \, dV = \iiint_{0}^{\pi/2} \int_{0}^{\pi} \int_{0}^{2\pi} 5r^2 \sin^2 \phi \cos^2 \theta \, r^2 \sin \phi \, dr \, d\phi \, d\theta = \frac{2\pi}{3} \]

The last iterated integral is equal to

\[ 5 \frac{2\pi}{3} \]

\[ \int_{0}^{\pi/2} \sin \phi \, d\phi = \frac{2\pi}{3} \]

\[ \int_{0}^{\pi/2} \sin^3 \phi \, d\phi = \int_{0}^{\pi/2} \sin \phi \, d\phi = \frac{\pi}{2} \]

\[ \left[ \cos \frac{\pi}{2} - \cos 0 \right] = -1 \]

\[ \frac{1}{3} \left[ \cos^3 \frac{\pi}{2} - \cos^3 0 \right] = -\frac{1}{3} \]

\[ \int_{0}^{\pi} \cos 2\theta \, d\theta = \frac{1}{2} \int_{0}^{2\pi} \cos 2\theta \, d\theta + \frac{1}{2} \int_{0}^{\pi} 1 \, d\theta = \pi. \]
10. (3 points) Evaluate the line integral

\[ \int_C \vec{F} \cdot dr, \]

for the vector field \( \vec{F}(x, y, z) = -y \cdot \hat{i} + x \cdot \hat{j} - z \cdot \hat{k}, \) where curve \( C \) is the boundary of the triangle with vertices \((0, 0, 5), (2, 0, 1)\) and \((0, 3, 0)\) traced in this order.

By Stokes' theorem, we have

\[ \int_C \vec{F} \cdot dr = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds. \]

Now, \( \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \ 0 & 1 \ \end{vmatrix} = <0, 0, 2> \]

A normal vector \( \hat{n} \) to the surface \( S \) is given by

\[ \hat{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -4 \\ 0 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \ 0 & 3 \ \end{vmatrix} = \begin{vmatrix} 1 & 0 \ 0 & 1 \ \end{vmatrix} = <12, 6, 6>. \]

The surface \( S \) lies in the plane of equation \( 12x + 6y + z - 5 = 0 \) or \( z = 5 - 2x - y \). Using \( x \) and \( y \) as parameters we define \( S \) by \( g(x, y) = <x, y, 5 - 2x - y> \), where \( (x, y) \in D \) and \( D \) is the triangle with vertices \((0, 0), (2, 0), \) and \((0, 3)\) in the \( xy \)-plane. We have

\[ g_x(x, y) = <1, 0, -2> \] and \( g_y(x, y) = <0, 1, -1> \). Moreover,

\[ g_x \times g_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \ \end{vmatrix} = \begin{vmatrix} 1 & 0 \ 0 & 1 \ \end{vmatrix} = <2, 1, 1>. \]

Then, \( \int_C \vec{F} \cdot dr = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S <0, 0, 2> \cdot <2, 1, 1> \, dA = 2 \iint_S dA = 2 \times 3 = 6. \)
11. (3 points) (BONUS!) (i) State carefully the divergence theorem.

(ii) Suppose that $S$ is the boundary of $E$ as in the divergence theorem. Prove that if functions $f$ and $g$ have continuous partial derivatives, then we have

$$
\int \int_S (f \text{grad} g - g \text{grad} f) \cdot d\mathbf{S} = \int \int \int_E (f \Delta g - g \Delta f) dV.
$$

**The divergence theorem.**

Let $E$ be a solid with piecewise smooth boundary that has positive (outward) orientation. Let $\mathbf{F}$ be a vector field defined in a domain that contains $E$. Then

$$
\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int \int_E \text{div} \mathbf{F} dV.
$$

Clearly, $f \text{grad} g - g \text{grad} f$ is a vector field, so by the divergence theorem, we obtain

$$
\int \int_S (f \text{grad} g - g \text{grad} f) \cdot d\mathbf{S} = \int \int \int_E \text{div}(f \text{grad} g - g \text{grad} f) dV.
$$

On the other hand, $\text{div}(f \text{grad} g - g \text{grad} f) = \text{div} < f \text{grad} x - g \text{grad} x, f \text{grad} y - g \text{grad} y, f \text{grad} z - g \text{grad} z > = (f \text{grad} x - g \text{grad} x)_x + (f \text{grad} y - g \text{grad} y)_y + (f \text{grad} z - g \text{grad} z)_z = f \text{grad} x + f \text{grad} y - g \text{grad} x + f \text{grad} y + f \text{grad} z - g \text{grad} y + f \text{grad} z - g \text{grad} z = f \text{grad} x + f \text{grad} y + f \text{grad} z - g \text{grad} x - g \text{grad} y - g \text{grad} z = f(\text{grad} x + \text{grad} y + \text{grad} z) - g(\text{grad} x + \text{grad} y + \text{grad} z) = f \Delta g - g \Delta f.$

□