Problem. (Rudin's book)

Suppose $f$ is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and $\int_a^b f(x)^2 \, dx = 1$. Note that

$$\int_a^b f(x) f(x) \, dx = -\frac{1}{2},$$

and that

$$\int_a^b (f'(x))^2 \, dx \cdot \int_a^b f'(x)^2 \, dx \geq \frac{1}{4}.$$

Solution.

(a) We use integration by parts:

$$\int_a^b f(x)^2 \, dx = \int_a^b x^2 f(x)^2 \, dx = \int_a^b f(x)^2 \, dx - \int_a^b f(x)^2 \, dx = 0 - 2 \int_a^b x f(x) f(x) \, dx$$

Thus

$$\int_a^b x f(x) f(x) \, dx = -\frac{1}{2}.$$

(b) We use the Cauchy-Schwarz's inequality:

$$\left| \int_a^b x f(x) f(x) \, dx \right| \leq \left( \int_a^b (x f(x))^2 \, dx \right)^{\frac{1}{2}} \left( \int_a^b (f(x))^2 \, dx \right)^{\frac{1}{2}}.$$

Squaring the above inequality, we get:

$$\frac{1}{4} \leq \left( \int_a^b (x f(x))^2 \, dx \right) \cdot \left( \int_a^b (f(x))^2 \, dx \right).$$
Problem: (Pittsburgh Helim Exam, 2011)
Assume \( f \in C^4([a,b]) \) and \( f''(a) = 0 \). Show that
\[
4 \int_a^b x^2 (f''(x))^2 \, dx \geq \int_a^b f''(x) \, dx.
\]

**Solution 1.**

\[
\int_a^b f''(x) \, dx = \int_a^b x \cdot f''(x) \, dx = \left. x f''(x) \right|_a^b - \int_a^b x^2 f'''(x) \, dx.
\]

Now, we have:

\[
\left| -2 \int_a^b x f''(x) \, dx \right| = 2 \int_a^b f''(x) \cdot f'''(x) \, dx \leq
\]

\[
2 \left( \int_a^b f''(x) \, dx \right)^2 \cdot \left( \int_a^b (f'''(x))^2 \, dx \right)^{1/2},
\]

and thus, we get

\[
\left( \int_a^b f''(x) \, dx \right)^2 \leq 2 \left( \int_a^b (f''(x))^2 \, dx \right)^{1/2} \Leftarrow \Rightarrow \]

\[
\int_a^b f''(x) \, dx \leq 4 \int_a^b x^2 (f''(x))^2 \, dx.
\]

**Solution 2.**

Again, as in the previous solution, we have:

\[
\int_a^b f''(x) \, dx = \int_a^b 2x f''(x) \, dx.
\]

We have:

\[
0 \leq \int_a^b (f''(x) + x f''(x)) \, dx = \int_a^b (f''(x))^2 + 2x f''(x) f'(x) + x^2 f'''(x) \, dx
\]

\[
= \int_a^b (f'(x))^2 \, dx + 2 \int_a^b x f''(x) f'(x) \, dx + \int_a^b x^2 f'''(x) \, dx.
\]

???
We start with the integral:

\[ 0 \leq \int_0^1 (2x^2 f(x) + f(x))^2 \, dx = \int_0^1 4x^2 f(x)^2 \, dx + 4 \int_0^1 x f'(x) f(x) \, dx + \int_0^1 f'^2(x) \, dx \geq 0. \]

But, this is equivalent to:

\[ 4 \int_0^1 x^2 f(x)^2 \, dx - 2 \int_0^1 f(x) f'(x) \, dx + \int_0^1 f'^2(x) \, dx \geq 0, \]

which is equivalent to

\[ 4 \int_0^1 x^2 f(x)^2 \, dx \geq \int_0^1 f'^2(x) \, dx. \]
Problem.

Let \( f, g \in C^1([a, b]) \) such that \( \int_a^b f^2 \, dx = \int_a^b g^2 \, dx = 1 \) and \( f(a) = g(b) = 0 \). Prove that

\[
\left| \int_a^b f g \, dx \right| \leq \left( \int_a^b f^2 \, dx \right)^{1/2} \left( \int_a^b g^2 \, dx \right)^{1/2} + \left( \int_a^b f g^2 \, dx \right)^{1/2} \left( \int_a^b f^2 g \, dx \right)^{1/2}
\]

Solution.

One can rewrite the inequality as:

\[
\left( \int_a^b f^2 \, dx \right)^{1/2} \left( \int_a^b g^2 \, dx \right)^{1/2} + \left( \int_a^b f g^2 \, dx \right)^{1/2} \left( \int_a^b f^2 g \, dx \right)^{1/2} \geq \left| \int_a^b f g \, dx \right|
\]

By the Cauchy-Schwarz's inequality, we have:

1. \( \left( \int_a^b f^2 \, dx \right)^{1/2} \left( \int_a^b g^2 \, dx \right)^{1/2} \geq \left| \int_a^b f g \, dx \right| \)

2. \( \left( \int_a^b f g^2 \, dx \right)^{1/2} \left( \int_a^b f^2 g \, dx \right)^{1/2} \geq \left| \int_a^b f g \, dx \right| \)

By adding up (1) and (2), we get:

\[
\left( \int_a^b f^2 \, dx \right)^{1/2} \left( \int_a^b g^2 \, dx \right)^{1/2} + \left( \int_a^b f g^2 \, dx \right)^{1/2} \left( \int_a^b f^2 g \, dx \right)^{1/2} \geq \left| \int_a^b f g \, dx \right| + \left| \int_a^b f^2 g \, dx \right| + \left| \int_a^b f g^2 \, dx \right|
\]

\[
= \left| \int_a^b f g \, dx \right| = \left| \int_a^b f g \, dx \right|
\]

integrating by parts,

\[
= \left| \int_a^b f g \, dx \right|
\]
Given \( \mu \in C([0,1]) \), define \( f : [0,1] \to \mathbb{R} \) by
\[
 f(x) = \begin{cases} 
 \int_0^x u(t) dt, & \text{if } x \in (0,1] \\
 \mu(0), & \text{if } x = 0
\end{cases}
\]
Prove that
\[
\left( \int_0^1 (f(x))^2 \, dx \right)^{1/2} \leq 2 \left( \int_0^1 (\mu(x))^2 \, dx \right)^{1/2}
\]

**Solution.**
Let us rewrite the inequality as
\[
\left( \int_0^1 (\frac{1}{x} \int_0^x u(t) \, dt)^2 \, dx \right)^{1/2} \leq 2 \left( \int_0^1 (\mu(x))^2 \, dx \right)^{1/2}
\]
By ignoring the above inequality, we obtain:
\[
\int_0^1 (\frac{1}{x} \int_0^x u(t) \, dt)^2 \, dx \leq 4 \int_0^1 (\mu(x))^2 \, dx.
\]
We have:
\[
\int_0^1 \frac{1}{x} \left( \int_0^x u(t) \, dt \right)^2 \, dx = \int_0^1 \left( \int_0^x u(t) \, dt \right)^2 \frac{1}{x} \, dx
\]
By parts
\[
= - \int_0^1 \left. \frac{1}{x} \left( \int_0^x u(t) \, dt \right)^2 \right|_0^1 + \int_0^1 \frac{1}{x} \left( \int_0^x u(t) \, dt \right)^2 \, dx
\]
\[
= - \int_0^1 \left( \int_0^x u(t) \, dt \right)^2 \, dx + \int_0^1 \frac{1}{x} \int_0^x u(t) \, dt \, dx
\]
\[
= - \int_0^1 \left( \int_0^x u(t) \, dt \right)^2 \, dx + \int_0^1 \frac{1}{x} \int_0^x u(t) \, dt \, dx
\]
\[
\leq 0
\]

**WARNING**
\[
\lim_{x \to 0} \frac{1}{x} \left( \int_0^x u(t) \, dt \right)^2
\]
\[ \leq 2 \int_0^1 \mu(x) \cdot (\frac{1}{x} \int_0^x \eta(t) dt) dx. \] On the other hand, by the Cauchy-Schwarz's inequality, we obtain:
\[ \int_0^1 \mu(x) \cdot (\frac{1}{x} \int_0^x \eta(t) dt) dx \leq (\int_0^1 \mu(x) dx)^{\frac{1}{2}} \cdot (\int_0^1 (\frac{1}{x} \int_0^x \eta(t) dt)^2 dx)^{\frac{1}{2}}. \]
Thus, we obtain:
\[ \int_0^1 (\frac{1}{x} \int_0^x \eta(t) dt)^2 dx \leq 2 \left( \int_0^1 \mu(x) dx \right)^{\frac{1}{2}} \cdot (\int_0^1 (\frac{1}{x} \int_0^x \eta(t) dt)^2 dx)^{\frac{1}{2}}. \]
\[ \Rightarrow \left( \int_0^1 (\frac{1}{x} \int_0^x \eta(t) dt)^2 dx \right)^{\frac{1}{2}} \leq 2 \left( \int_0^1 \mu(x) dx \right)^{\frac{1}{2}}. \]

**Remark.**

The map \( Hf(x) = \frac{1}{x} \int_0^x \eta(t) dt \) is called the *Hardy operator*. Basically, our inequality states that:
\[ \| H \mu \|_{L^2(\mathbb{R}^+)} \leq C \cdot \| \mu \|_{L^2(\mathbb{R}^+)} \]
where \( C = 2 \).

This is a particular case of the so-called *Hardy's inequality*:
\[ \| H \mu \|_{L^1([0,1])} \leq C \cdot \| \mu \|_{L^p([0,1])} \]
where \( C = \left( \frac{p}{p-1} \right)^{\frac{1}{p}} \).
Problem. (Putnam Competition 1964 & Berkeley Preliminary Exam)

Let $0 \leq a \leq 1$. Find all continuous functions $f: [0, 1] \rightarrow (0, \infty)$ such that

\[ \int_0^1 f(x)dx = 1, \int_0^1 xf(x)dx = a, \int_0^1 x^2 f(x)dx = a^2, \]

Solution.

\[ \int_0^1 f(x)dx = 1 \Rightarrow \int_0^1 x f(x)dx = \int_0^1 x^2 f(x)dx = a. \]

\[ \int_0^1 x f(x)dx = a \Rightarrow -2a \int_0^1 (x f(x))dx = -2a. \]

\[ \int_0^1 -2a x f(x)dx = -2a. \]

And finally, we have:

\[ \int_0^1 x^2 f(x)dx = a^2/1 \Rightarrow \int_0^1 x^2 f(x)dx = a. \]

Adding these 3 equalities, we get:

\[ 0 (x-a) = \int_0^1 f(x) (x-a)^2 dx \Rightarrow f=0, \text{false!} \]

Alternative solution.

By Cauchy-Schwarz's inequality, we have:

\[ \left( \int_0^1 \sqrt{f(x)} \right)^2 \cdot \left( \int_0^1 \sqrt{f(x)} \right)^2 \geq \left( \int_0^1 x f(x)dx \right)^2 \]

\[ \Rightarrow \frac{a^2}{1} \geq a^2. \]

Equality holds true iff $x\sqrt{f(x)} = \sqrt{f(x)} \Rightarrow f(x) = 0,$ false!

There is not such function.
Problem. (Elemente der Mathematik, 1981)
Let \( f \in C^2([a,b]) \) such that \( f(a) = f(b) = 0 \) Show that
\[
\int_a^b (f'(x))^2 \, dx \geq 12 \left( \int_a^b f(x) \, dx \right)^2.
\]

Solution.
Let us rewrite the given inequality like this:
\[
\frac{1}{12} \int_a^b (f'(x))^2 \, dx \geq \left( \int_a^b f(x) \, dx \right)^2,
\]

or even more:
\[
\int_a^b (x - \frac{1}{2})^2 \, dx = \int_a^b (x^2 - x + \frac{1}{4}) \, dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{4}.
\]
Thus, one can rewrite the inequality as:
\[
\int_a^b (x - \frac{1}{2})^2 \, dx \cdot \int_a^b (f'(x))^2 \, dx \geq \left( \int_a^b f(x) \, dx \right)^2.
\]

By the Cauchy-Schwarz's inequality, we have:
\[
\int_a^b (x - \frac{1}{2})^2 \, dx \cdot \int_a^b (f'(x))^2 \, dx \geq \left( \int_a^b (x - \frac{1}{2}) f(x) \, dx \right)^2.
\]

Now, we have:
\[
\left( \int_a^b (x - \frac{1}{2}) f(x) \, dx \right)^2 = \left( \int_a^b x f(x) \, dx - \frac{1}{2} \int_a^b f(x) \, dx \right)^2 = \left( x f(x) \right)_a^b - \int_a^b f(x) \, dx - \frac{1}{2} (f(a) - f(b)) \right)^2 = \left( f(a) - \int_a^b f(x) \, dx \right)^2 \geq \left( \int_a^b f(x) \, dx \right)^2.
\]
The conclusion follows immediately. \( \square \)
Problem. (Wirtinger's inequality - weak form)

Let \( f \in C^1([0,1]) \) such that \( f(0) = f(1) = 0 \). Show that

\[
\int_0^1 (f'(x))^2 \, dx \geq \frac{\pi^2}{4} \int_0^1 f(x)^2 \, dx.
\]

Solution.

With the given hypothesis, the function \( f(x) \cot nx \)
has lateral limit.

Therefore, the following integral can be computed by limits:

\[
\lim_{x \to 0^+} \frac{2}{nx^2} f(x) \cot nx = \lim_{x \to 0^+} \frac{\frac{2}{nx^2} f(x) \cot nx}{\frac{2}{nx^2} \sin nx} = \lim_{x \to 0^+} \frac{2}{nx^2} \cot nx = 0.
\]

\[
2\pi \int_0^1 f(x) \cot nx \, dx = 2\pi \left[ \frac{1}{n} f(x) \cot nx \right]_0^1 + \pi^2 \int_0^1 f(x)^2 \, dx \cot^2 nx.
\]

We have

\[
\int_0^1 \left[ (f'(x))^2 - \pi^2 f(x)^2 \right] \, dx = \int_0^1 \left[ f'(x) - \pi \cot nx \cdot f(x) \right]^2 \, dx \geq 0.
\]

The equality holds iff \( \frac{f'(x)}{f(x)} = \frac{\pi \cot nx}{\sin nx} \), so \( f(x) = C \sin nx \). \( \Box \)
Problem (Ohio State Prelim, 2005).

Let \( f: [0,1] \rightarrow \mathbb{R} \) be a continuous function such that
\[ \star \quad \int_0^1 fg(x) \, dx = 0, \]
for each continuously differentiable function \( g: [0,1] \rightarrow \mathbb{R} \) satisfying \( g(0) = 0 = g(1) \). Prove that \( f \) must be a constant function.

Solution.

If we plug in \( f = c \), we get
\[ \int_0^1 cg(x) \, dx = 0 \Rightarrow c \int_0^1 g(x) \, dx = c (g(1) - g(0)) = 0. \]
Now, what if we plug in \( f = -c \) instead of \( f \) in \( \star \). We get:
\[ 0 = \int_0^1 (f(x) - c) g(x) \, dx \Rightarrow 0 = \int_0^1 f(x) g(x) \, dx - c \int_0^1 g(x) \, dx. \]

In fact, we shall prove that \( f(x) = \int_0^x f(t) \, dt \).

Now, let's consider the function \( F: [0,1] \rightarrow \mathbb{R} \),
\[ F(x) = \int_0^x f(t) \, dt. \]
This tells us that \( F \) is differentiable with \( g(0) = g(1) = 0 \), we have:
\[ \int_0^1 F(x) g(x) \, dx = \int_0^1 (f(x) - \int_0^x f(t) \, dt) g(x) \, dx = \]
\[ = \int_0^1 f(x) g(x) \, dx - \int_0^1 \int_0^x f(t) \, dt g(x) \, dx = 0. \]

Let \( g(x) = \int_0^x f(t) \, dt \). Clearly, this is differentiable
and \( g(0) = 0 \) and
\[ g(1) = \int_0^1 f(x) - \int_0^1 f(t) \, dt \, dx = \int_0^1 f(x) \, dx - \int_0^1 f(t) \, dt \, dx = 0. \]

Thus it follows that
\[ 0 = \int_0^1 F(x) g(x) \, dx = \int_0^1 F(x) F(x) \, dx \Rightarrow F(x) = 0, \]
\( \forall x \in [0,1] \Rightarrow f(x) = \int_0^x f(t) \, dt. \)
Second solution.

Let $c = \int \phi(x) \, dx$ and consider the function

$g(x) = F(x) - c \cdot x$, where $F(x) = \int_0^x f(t) \, dt$. Clearly $g(x)$
is continuously differentiable with $g(x) = g'(x) = 0$. By
assumption, we have that

$$0 = \int_0^1 g(x) \, dx = \int_0^1 F(x)(f(x) - c) \, dx = \int_0^1 f(x) \, dx - c \int_0^1 f(x) \, dx,$$

so

$$\int_0^1 f(x) \, dx = (\int_0^1 f(x) \, dx)^2 \quad \Rightarrow \quad f(x) = \lambda \cdot 1, \quad \lambda = \text{constant}$$

Equality in Cauchy-Schwarz's

inequality

From the Cauchy-Schwarz's inequality, we have

\[(C-S): \int_0^1 f^2(x) \, dx \cdot \int_0^1 g^2(x) \, dx \geq (\int_0^1 f(x) g(x) \, dx)^2.\]

For $g(x) = 1, \forall x \in [0, 1]$, (C-S) is equivalent to

$$\int_0^1 f^2(x) \, dx \geq (\int_0^1 f(x) \, dx)^2,$$

with equality iff $f(x) = \lambda \cdot 1, \quad \lambda = \text{constant}$. 

\[\square\]
Problem (Ohio State Prelims, 2004).

Let the function $y$ be continuous on $[0,1]$ with
\[ \int_0^1 y(x) \, dx = 0 \quad \text{and} \quad \int_0^1 x^2 y(x) \, dx = 1. \]

Prove that $|y(x)| \geq 4$ for some $x \in [0,1]$.

Solution.

Suppose by contradiction that $|y(x)| < 4$, for all $x \in [0,1]$. Consider the integral
\[ 1 = \left| \int_0^{1/2} (x - 1/2) y(x) \, dx \right| < \int_0^{1/2} (x - 1/2) \, dx = 1/4. \]

\[ \leq 4 \cdot \int_0^{1/2} |x - 1/2| \, dx. \]

Now, we have:
\[ |x - 1/2| = \begin{cases} 
+ (x - 1/2), & \text{if } x \geq 1/2 \\
-(x - 1/2), & \text{if } x \leq 1/2
\end{cases} \]

\[ = 4 \cdot \left( \int_0^{1/2} (x - 1/2) \, dx + \int_0^{1/2} (x - 1/2) \, dx \right) \]

\[ = 4 \left( -\frac{1}{2} \int_0^{1/2} x \, dx - \frac{1}{2} \int_0^{1/2} x \, dx + \int_0^{1/2} x \, dx - \frac{1}{2} \int_0^{1/2} x \, dx \right) \]

\[ = 4 \left( -\frac{x^2}{2} \bigg|_0^{1/2} - \frac{1}{2} x \bigg|_0^{1/2} + \frac{x^2}{2} \bigg|_0^{1/2} - \frac{1}{2} x \bigg|_0^{1/2} \right) \]

\[ = 4 \left( -\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - \frac{1}{8} - \frac{1}{2} (1 - \frac{1}{2}) \right) \]

\[ = 4 \left( -\frac{1}{2} + \frac{1}{2} \right) = 1, \quad \text{contradiction}.

Remark. $y(x = ax + b =)$

\[ \int_0^1 y(x) \, dx = \int_0^1 (ax + b) \, dx = \frac{a}{2} + b = 0. \]

\[ \int_0^1 x \cdot y(x) \, dx = \int_0^1 (ax^2 + bx) \, dx = \frac{a}{3} + \frac{b}{2} = 1. \]
Solution 2.
Suppose by contradiction the contrary, that $|f(x)| \leq 4$ for all $x \in [0,1]$. Now, define

$$f(x) = \int_0^x 4t\,dt.$$ 

Clearly $f$ is differentiable and $f(0) = f(1) = 0$ and $f(\alpha) = \int_0^\alpha 4x\,dx = 0$. The second condition implies that $\int_0^1 f(x)\,dx = 1$, since by integrating by parts, we have

$$1 = \int_0^1 \alpha f(\alpha)\,d\alpha = \int_0^1 f(x)\,dx - \int_0^1 f\,dx = \int_0^1 f\,dx.$$ 

Now the idea is the following:
Since $|f'| = |4x| < 4$, and $f(0) = f(1) = 0$, the graph of $y = f(x)$ must be enclosed by four lines of "maximal" slope, namely, $y = \pm 4x$ and $y = \pm 8\,\pm 4x$. But then the area under $y = f(x)$ must be strictly less than 1, contrary to $\int_0^1 f(x)\,dx = 1$. Let's be more precise:

We claim that

$$|f(x)| < 4|x|,$$

and

$$|f(x)| < 8 - 4|x|$$

First, suppose the contrary, $|f(x)| \geq 4|x|$ for some $x \in [0,\frac{1}{2}]$. By the mean value theorem, there exists $\eta \in (0,\frac{1}{2})$ such that

$$|f(\eta)| = \left| \frac{f(\eta) - f(0)}{\eta - 0} \right| = \left| \frac{f(\eta)}{\eta} \right| = \left| \frac{f(\eta)}{\eta} \right| > 4,$$

in contradiction with $|f| < 4$. A similar argument works for the second part of our claim. Now recall if $\phi$ is a continuous non-negative function on $[0,1]$ and $\int_0^1 \phi(x)\,dx = 0$, then $\phi \equiv 0$. This implies that

$g_1, g_2, h$ are continuous functions on $[0,1]$ such that

$$g_1(x), g_2(x), h(x) \geq 0$$
g(x) < h(x) for some \( x \in [0,1] \), then \( \int_0^1 g(x) \, dx < \int_0^1 h(x) \, dx \). In particular, we have

\[
\int_0^{1/2} f(x) \, dx < \int_0^{1/2} 4x \, dx, \quad \int_0^{1/2} f(x) \, dx < \int_0^{1/2} (8-4x) \, dx.
\]

Therefore, we have:

\[
1 = \left| \int_0^1 f(x) \, dx \right| \leq \int_0^{1/2} f(x) \, dx < \int_0^{1/2} 4x \, dx + \int_0^{1/2} (8-4x) \, dx = \frac{1}{2} + \frac{1}{2} = 1;
\]

Remark.

The following generalization also holds true:

Let \( g \) be a continuous function on \([0,1]\) such that

\[
\int_0^1 g(x) \, dx = 0 \quad \text{and} \quad \int_0^1 x^k g(x) \, dx = 1, \text{ for some } k = 1, 2, \ldots, n.
\]

Show that there exists \( x_0 \in [0,1] \) such that

\[
|g(x_0)| \geq 2^n(n+1).
\]

The idea is to consider the integral

\[
|I| = \left| \int_0^1 (x-\frac{1}{2})^n g(x) \, dx \right| = \ldots \quad \text{and proceed exactly like in the first solution.}
\]