Problem.

Let \((a_n)_{n=1}^{\infty}\) be a decreasing sequence of nonnegative real numbers. Suppose that \(\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{n}}\) is convergent. Prove that \(\sum_{n=1}^{\infty} a_n\) is also convergent.

Solution.

Since the series \(\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{n}}\) is convergent and \(a_n \geq 0\) for all \(n \in \mathbb{N}\), it follows that

\(S_n = \sum_{j=1}^{n} \sqrt{\frac{a_j}{j}}\) is bounded.

i.e., \(S_n = \sqrt{\frac{a_1}{1}} + \sqrt{\frac{a_2}{2}} + \ldots + \sqrt{\frac{a_n}{n}} \leq M\). and since \(a_n\) is decreasing, we have:

\(M \geq S_n \geq n \sqrt{\frac{a_n}{n}} = \sqrt{a_n}\) for all \(n \geq 1\).

Now, since \(\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{n}}\) converges, we have that

\(M \cdot \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} n \cdot \frac{\sqrt{a_n}}{n}\) is also convergent.

Since \(1 \cdot \sqrt{a_1} = a_1 = \sqrt{a_1} \cdot \sqrt{a_1} \leq \sqrt{a_1} \cdot \sqrt{a_2} \leq \ldots \leq \sqrt{a_1} \cdot \sqrt{a_n}\) by the comparison test, it follows that \(\sum_{n=1}^{\infty} a_n\) converges.
Problem. (Berkeley Prelim Exam, 1985)

Define the Riemann zeta function,

\[ \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}. \]

Prove that \( \zeta(x) \) is defined and has continuous derivatives of all orders in the interval \( 1 < x < \infty \).

Solution.

Let \( \sigma > 1 \). It suffices to show that \( \zeta(x) \) is defined and has continuous derivatives for \( x \geq \sigma \). It is well-known that the series \( \sum_{n=1}^{\infty} \frac{1}{n^x} \) is convergent for such \( x \).

As \( \frac{1}{n^x} \leq \frac{1}{n^{-\sigma}} \), it follows by the Weierstrass M-test that the series converges uniformly, so \( \zeta \) is a continuous function. To see that it has continuous derivatives of all orders, we formally differentiate the series \( k \) times, getting

\[ \sum_{n=2}^{\infty} \frac{(-\log n)^k}{n^x}. \]

It is enough to show that this series converges uniformly in \( x \). Since

\[ \left| \frac{(-\log n)^k}{n^x} \right| \leq \frac{\left| \log n \right|^k}{n^x}, \]

by the Weierstrass M-test, it will suffice to show that the series

\[ \sum_{n=2}^{\infty} \frac{\left| \log n \right|^k}{n^x} \]

converges.
But \( \frac{(\log n)^k}{n^\sigma} = o \left( \frac{1}{n^{\sigma-\epsilon}} \right) \) (\( n \to \infty \))
for any positive \( \epsilon \). As
\[
\sum_{n=1}^{\infty} \left( \frac{1}{n^{\sigma-\epsilon}} \right)
\]
converges for \( \sigma - \epsilon > 1 \), we are done. \( \square \)
Problem. (Hajlave's old exams)
Let \( f : [0, +\infty) \to (0, +\infty) \) be a continuous function. Prove that for every \( x > 0 \) the following inequality is true:

\[
\int_0^x t^2 f(t) \, dt \cdot \int_0^x f(t) \, dt \geq \left( \int_0^x t f(t) \, dt \right)^2.
\]

Solution.
This is just Cauchy–Schwarz's inequality applied for \( t \mapsto tf(t) \) and \( t \mapsto \sqrt{f(t)} \). \( \Box \)

Another solution.
Let us start with the following double integral

\[
0 \leq \int_0^x (x + s - 2\sqrt{st}) f(t) f(s) \, dt \, ds = \int_0^x (t f(t) f(s) + s f(t) f(s)) \, dt \, ds.
\]
Problem. (Berkeley Preliminary Exam 2004)

A $C^2$ function $y(x)$ for $0 \leq x \leq 1$, a positive continuous function $a(x)$ for $0 \leq x \leq 1$, and a real number $\lambda$ satisfy

$$\begin{cases} y''(x) + 2 \cdot a(x) \cdot y(x) = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Suppose $y(x)$ is not identically zero. Prove that $\lambda > 0$.

Solution.

Multiply the ODE by $y(x)$ and integrate by parts and we get:

$$\int_0^1 2 \cdot a(x) y'^2 \, dx = \int_0^1 y(x) y'' dx = \left[ y(x) y'(x) \right]_0^1 - \int_0^1 y'^2 \, dx$$

$$= \int_0^1 (y'^2) \, dx > 0$$

Since if $y'$ were identically zero on $[0,1]$, the $y$ would be constant on $[0,1]$, making $y \equiv 0$ since $y(0) = 0$. Since $a > 0$ and $y$ is not identically zero, we also have $\int_1^0 y'^2 \, dx > 0$. Thus $x$ is a ratio of positive numbers, so $\lambda > 0$. \[\square\]
Problem. (Wirtinger's inequality)

Let \( f \) be a twice differentiable function real valued on \([0, 2\pi]\), with

\[
\int_0^{2\pi} f(x) \, dx = 0 = f(2\pi) - f(0).
\]

Show that

\[
\int_0^{2\pi} f^2(x) \, dx \leq \int_0^{2\pi} (f'(x))^2 \, dx.
\]

Selection.

Consider the Fourier series of \( f \),

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).
\]

We have \( a_0 = 0 \). By Parseval's identity, we have:

\[
\int_0^{2\pi} f^2(x) \, dx = \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \int_0^{2\pi} (f'(x))^2 \, dx.
\]

\[
\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.
\]

\[
\sum_{n=-\infty}^{\infty} c_n \cdot c_m = \sum_{n=-\infty}^{\infty} (a_n \cdot a_m \cos(nx) + b_n \cdot b_m \sin(nx)) \cos(mx) + \sin(mx)(a_n \cdot b_m \cos(nx) + b_n \cdot a_m \sin(nx))
\]
Problem. (Berkeley Preliminary Exam)

Let $f$ be a real valued continuous function on $[0, \infty)$ such that

$$\lim_{x \to \infty} (f(x) + \int_0^x f(t) \, dt)$$

exists. Prove that $\lim_{x \to \infty} f(x) = 0$.

Solution:

Let us consider the function $g : [0, \infty) \to \mathbb{R}$,

$$g(x) = f(x) + \int_0^x f(t) \, dt.$$

We prove that:

1. $\lim_{x \to \infty} g(x) = 0$.

Suppose by contradiction that $\lim_{x \to \infty} g(x) > 0$. This means that there are $\varepsilon, x_0 > 0$ such that $f(x) > \varepsilon$ for $x > x_0$. Then, we have

$$g(x) = f(x) + \int_0^x f(t) \, dt = f(x) + \int_0^{x_0} f(t) \, dt + \int_{x_0}^x f(t) \, dt$$

$$\geq \varepsilon + \int_{x_0}^x f(t) \, dt + \varepsilon (x - x_0),$$

which is a contradiction as $x \to \infty$.

2. $\limsup_{x \to \infty} f(x) \geq 0$.

Apply previous claim to the function $-f$.

3. $\limsup_{x \to \infty} f(x) \leq 0$. 

-1
Assume by contradiction it is not true. Then for some $\epsilon > 0$ there is a sequence $x_1, x_2, \ldots$, tending to $\infty$ such that $f(x_n) > \epsilon$ for all $n$. By 1°, the function $f$ assumes values $\leq \frac{\epsilon}{2}$ for arbitrarily large values of its argument. Then after deleting finitely many $x_n$'s, we can find another sequence $y_1, y_2, \ldots$, tending to $\infty$ such that $y_n < x_n$ for all $n$ and $f(y_n) \leq \frac{\epsilon}{2}$ for all $n$. Let $x_{\text{min}}$ be the largest number in $[y_1, x_{\infty}]$ where $f$ takes the values $\frac{\epsilon}{2}$ (it exists by the Intermediate Value Theorem). Then

$$g(x_n) - g(x_{\text{min}}) = f(x_n) - f(x_{\text{min}}) + \int_{x_{\text{min}}}^{x_n} \frac{f(t)}{t} \, dt$$

$$\geq \epsilon - \frac{\epsilon}{2} + \int_{x_{\text{min}}}^{x_n} \frac{\epsilon}{2} \, dt$$

$$\geq \frac{\epsilon}{2},$$

which contradicts the existence of $\lim_{x \to \infty} g(x)$.

1°. $\lim_{x \to \infty} f(x) > 0$.

Apply 3° for $-f$. \qed