Another look at some new Cauchy–Schwarz type inner product inequalities

Cezar Lupu a,b,* , Dan Schwarz c

a University of Pittsburgh, Department of Mathematics, Pittsburgh, PA 15260, USA
b University of Craiova, Department of Mathematics, Str. Alexandru Ioan Cuza 13, RO–200585 Craiova, Romania
c Intuitext Sofwin Group, D. Pompeiu St. 10A, RO–020337 Bucharest, Romania

A R T I C L E   I N F O
Keywords:
Inner product space
Cauchy–Schwarz inequality
Gram determinant
Buzano’s inequality
Integral inequalities
Discrete inequalities
Lagrange multipliers

A B S T R A C T
In this paper we take a refreshing look at a Cauchy–Schwarz type inner product inequalities. We also provide a Cauchy–Schwarz based proof and a refinement of Buzano’s inequality in inner product spaces. Some elementary applications such as trace inequalities for unitary matrices and discrete or integral inequalities are given.

1. Introduction

The theory of inner product inequalities plays a central role in many branches of mathematics like Linear Operators, Partial Differential Equations, Approximation & Optimization Theory, Information Theory or Statistics and many other fields. Such inequalities were the pioneering work of mathematicians like Cauchy, Minkovski, Holder, Hilbert, Hardy, Kantorovich and many others.

On the other hand, it is worth mentioning the contributions of Bellman, Boas, van der Corput, Ostrowski, Selberg, Enflo and Bombieri who have applied successfully in obtaining applications for Fourier and Mellin transforms, oscillatory integrals, approximation of polynomials or large sieve. All these results as well as their applications are covered in the monograph [4].

In [2], Buzano gave the following extension of the celebrated Cauchy–Schwarz’s inequality in a real or complex inner product space \((H; \langle \cdot, \cdot \rangle)\),

\[
|\langle a, c \rangle \langle c, b \rangle| \leq \frac{1}{2} (||a|| ||b|| + ||a, b||) \cdot ||c||^2
\]

for all \(a, b, c \in H\).

Clearly, when \(a = b\), the above inequality becomes the celebrated Cauchy–Schwarz’s inequality,

\[
|\langle a, c \rangle|^2 \leq ||a||^2 ||c||^2
\]

for all \(a, c \in H\).

As pointed out in [5], the original proof of Buzano is a little bit complicated. Buzano’s inequality also mentioned in [8] as an interesting generalization of the Cauchy–Schwarz’s inequality and its proof requires some facts about orthogonal
decomposition of a complete inner product space. Moreover, Fuji and Kubo [5] gave a simple proof by using the orthogonal projection on a subspace of an inner product space $H$ and Cauchy–Schwarz’s inequality.

The equality case in (1) holds if

$$ c = \begin{cases} \alpha \left( \frac{a}{|a|} + \frac{b}{|b|} \right), & \text{when } \langle a, b \rangle \neq 0, \\ \lambda \left( \frac{a}{|a|} + \beta \frac{b}{|b|} \right), & \text{when } \langle a, b \rangle = 0, \end{cases} $$

where $\alpha, \beta \in \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \}$. A detailed proof of Buzano’s inequality can also be found in [15], pp. 60–61.

For real inner product spaces, Richard [14] obtained the following stronger inequality

$$ |\langle a, c \rangle \langle c, b \rangle - \frac{1}{2} \langle a, b \rangle ||c||^2| \leq \frac{1}{2} ||a|| ||b|| ||c||^2. \tag{3} $$

$a, b, c \in H$.

Dragomir [4] showed that the above inequality is true with coefficients $\frac{1}{2}$ instead of $\frac{1}{3}$, where $\alpha$ is a non-zero number with $|1 - \alpha| = 1$. Also, Dragomir [4] obtained a refinement of Buzano’s inequality,

$$ |\langle a, c \rangle \langle c, b \rangle| \leq |\langle a, c \rangle \langle c, b \rangle - \frac{1}{2} \langle a, b \rangle ||c||^2| + \frac{1}{2} \langle a, b \rangle ||c||^2 \leq \frac{1}{2} (||a|| ||b|| + ||\langle a, b \rangle||)||c||^2, \tag{4} $$

$a, b, c \in H$.

In [6], Gavrea generalized Buzano’s inequality to nonnegative real exponents. In fact, he proved that

$$ |\langle a, x \rangle^\alpha \cdot |\langle x, b \rangle|^\beta | \leq t_1 t_2 ||x||^{\alpha + \beta}, $$

where $t_1$ and $t_2$ are expressed in terms of the constants $\alpha$ and $\beta$ and vectors $a, b$.

2. A Cauchy–Schwarz setting to prove Buzano’s inequality

In this section we prove Buzano’s inequality using a Krein type inequality which is deduced only from Cauchy–Schwarz’s inequality. As mentioned in [9], for any given two vectors $a, b \in \mathbb{C}^n \setminus \{0\}$, one can define

$$ \cos \Phi_{ab} = \frac{|\langle a, b \rangle|}{||a|| ||b||} $$

and

$$ \cos \Psi_{ab} = \frac{|\langle a, b \rangle|}{||a|| ||b||}. $$

In 1969, Krein proved the following inequality,

$$ \Phi_{ac} \leq \Phi_{ab} + \Phi_{bc} $$

for any $a, b, c \in \mathbb{C}^n$.

A proof of the above result can be found in [7,9]. Moreover, the main idea from the proof of Krein’s inequality is hidden in the following Krein-type inequality.

**Theorem 2.1.** For any vectors $a, b, c \in (H, \langle \cdot, \cdot \rangle)$ we have the following inequality

$$ ||a||^2 ||b||^2 + ||c||^2 (||a||^2 + ||b||^2) \leq ||a||^2 ||b||^2 ||c||^2 + 2 ||\langle a, b \rangle || \langle b, c \rangle \langle c, a \rangle. $$

*First proof.* We will give a proof using only Cauchy–Schwarz’s inequality. Without loss of generality, we assume $||c|| \neq 0$, otherwise the inequality is trivial. Now, for any complex scalar $\lambda$, by the Cauchy–Schwarz’s inequality, we have

$$ ||a - \lambda b||^2 ||c||^2 \geq ||a - \lambda b, c||^2 $$

for all $a, b, c \in H$. By expanding, this inequality is successively equivalent to

$$ ||\langle a, b \rangle^2 \lambda ||a, b||^2 + \lambda^2 ||b||^2 ||c||^2 \geq ||\langle a, a \rangle^2 - 2 \lambda ||\langle a, a \rangle ||b||^2 ||c||^2 + \lambda^2 ||a||^2 ||c||^2, $$

$$ ||b||^2 ||c||^2 - ||c, b||^2 ||c||^2 \lambda^2 + 2 \lambda^2 ||\langle a, b \rangle |||b||^2 ||c||^2 - ||\langle a, a \rangle ||b||^2 ||c||^2 ||c||^2 + ||b||^2 ||c||^2 - ||\langle a, a \rangle ||b||^2 ||c||^2 \geq 0 $$

for all complex scalars $\lambda$. The last inequality can be viewed as a quadratic polynomial in $\lambda$ with its dominant coefficient non-negative (by Cauchy–Schwarz’s inequality), and thus, the discriminant $\Delta$ must be negative,

$$ \Delta = 4 ||\langle a, b \rangle ||b||^2 ||c||^2 - ||\langle a, a \rangle ||b||^2 ||c||^2 \rangle^2 - 4 ||b||^2 ||c||^2 (||a||^2 ||c||^2 - ||\langle a, a \rangle ||b||^2 ||c||^2 \rangle \leq 0, $$

which is equivalent to
\[(a, b)^2 |b|^4 - 2(a, b)(|b, c| |c, a| |c| |c, a|)^2 < |a|^2 |b|^2 |c| |c, a| |c| |c, a| \]

and the desired inequality follows.

**Second proof.** Consider the following matrix

\[D(a, b) = \begin{bmatrix} |c|^2 & |\langle c, a \rangle| \\ |\langle b, c \rangle| & |\langle a, b \rangle| \end{bmatrix}.\]

We will prove that

\[|\det D(a, b)|^2 \leq |\det D(a, a)| \cdot |\det D(b, b)|.\]

Denote \(P(t) = \det(D(a, a))t^2 - 2\det(D(a, b))t + \det(D(b, b)).\) A simple calculation yields

\[P(t) = |\det D(at - b, at - b)| = \det \left[ \begin{bmatrix} |c|^2 & (ta - b, c) \\ (c, ta - b) & (ta, ta - b) \end{bmatrix} \right] = |c|^2 |at - b|^2 - |(at - b, c)|^2 \geq 0,\]

by Cauchy–Schwarz's inequality. This implies that the quadratic polynomial is nonnegative and thus, its discriminant must be negative,

\[\Delta_{P(t)} = |\det D(a, b)|^2 - |\det D(a, a)| \cdot |\det D(b, b)| \leq 0,\]

which is equivalent to

\[\left( |c|^2 |\langle a, b \rangle| - |\langle c, a \rangle||b, c| \right)^2 \leq (|c|^2 |a|^2 - |\langle c, a \rangle|^2)(|c|^2 |b|^2 - |b, c|^2)\]

and by expanding our inequality follows. \(\square\)

**Remark.** One can consider the Gram matrix

\[G(a, b, c) = \begin{bmatrix} |a|^2 & |\langle a, b \rangle| & |\langle a, c \rangle| \\ |\langle b, a \rangle| & |b|^2 & |\langle b, c \rangle| \\ |\langle c, a \rangle| & |\langle b, c \rangle| & |c|^2 \end{bmatrix}.\]

It is well-known that \(G(a, b, c)\) is positive semidefinite and thus \(\det G(a, b, c)\) is nonnegative, and finally this implies

\[\det G(a, b, c) = |a|^2 |b|^2 |c|^2 + 2(|a, b| ||b, c|| |c, a| - (|a|^2 |b, c|^2 + |b|^2 |c, a|^2 + |c|^2 |a, b|^2))\]

and the inequality follows. This idea appears also in [1].

As a consequence we have the following interesting.

**Corollary 2.2.** For any vectors \(a, b, c \in \langle H, \langle \cdot, \cdot \rangle \rangle\) we have the following inequality

\[|a|^2 |b|^2 - |\langle a, b \rangle|^2 \geq |a|^2 |b, 1|^2 + |b|^2 |a, 1|^2 - 2|a, 1||b, 1||\langle a, b \rangle|\]

**Proof.** Take \(c\) to be the unit vector in Theorem 1.2. \(\square\)

The above corollary served as application of several refinements of Heisenberg type inequalities related to uncertainty principle in the paper [13].

As a consequence of Theorem 1.2 we have the general integral version.

**Corollary 2.3.** Let \((H, \Sigma, \mu)\) be a positive measure space and let \(f, g, h \in L^2(H, \Sigma, \mu)\) the Hilbert space of complex-valued \(2 - \mu\) integrable functions defined on \(H\). Then

\[\left( \int_H f^2(t)d\mu(t) \right) \left( \int_H g(t)h(t)d\mu(t) \right)^2 + \left( \int_H g^2(t)d\mu(t) \right) \left( \int_H h(t)f(t)d\mu(t) \right)^2 + \left( \int_H h^2(t)d\mu(t) \right) \left( \int_H f(t)g(t)d\mu(t) \right)^2 \leq \left( \int_H f^2(t)d\mu(t) \right) \left( \int_H g^2(t)d\mu(t) \right)^2 + 2 \left( \int_H f^2(t)d\mu(t) \right) \left( \int_H f(t)g(t)d\mu(t) \right) \left( \int_H g(t)h(t)d\mu(t) \right).\]

It might be useful to see that out of Theorem 3.2 one can get the following discrete version.

**Corollary 2.4.** If \(p_i \geq 0, i = 1, 2, \ldots, n\) and \(a_i, b_i, c_i\) are complex numbers, then
Corollary 2.5. Let \(f, g, h\) be continuous real-valued functions on \([a, b]\). Then, we have the following inequality

\[
\left( \int_a^b f^2(t)dt \right) \left( \int_a^b g(t)h(t)dt \right) + \left( \int_a^b g^2(t)dt \right) \left( \int_a^b h(t)f(t)dt \right) + \left( \int_a^b h^2(t)dt \right) \left( \int_a^b f(t)g(t)dt \right) \\
\leq \left( \int_a^b f^2(t)dt \right) \left( \int_a^b g^2(t)dt \right) + \left( \int_a^b g^2(t)dt \right) \left( \int_a^b h^2(t)dt \right) + 2 \left( \int_a^b f(t)g(t)dt \right) \left( \int_a^b g(t)h(t)dt \right) \left( \int_a^b h(t)f(t)dt \right).
\]

The discrete real version of Corollary 2.4 is given by.

Corollary 2.6. Given real numbers \(a_i, b_i, c_i, 1 \leq i \leq n\), then

\[
\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) + \left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n a_i c_i \right) + \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i c_i \right) \\
\leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) + \left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) + 2 \left( \sum_{i=1}^n a_i b_i \right) \left( \sum_{i=1}^n b_i c_i \right) \left( \sum_{i=1}^n c_i a_i \right).
\]

In other words, if \(f, g, h\) are random variables of the space \(L^2(X, \mu)\), where \(\mu\) is a probability measure. Define the following covariance between \(f, g\) and \(h\) by

\[
cov_h(f, g) := E(fg)E(h^2) - E(fh)E(gh),
\]

where \(E(f) = \int_X fd\mu\) is the expectation of \(f\). The variance of \(f\) is given by

\[
\var(f) = \cov_h(f, f) = E(f^2) - E(f)^2.
\]

Thus, after some computations, Theorem 1.2 can be rewritten as the following variance–covariance inequality:

\[
|\cov_h(f, g)|^2 \leq \var(f) \var(g).
\]

In the case when \(h = 1\) we obtain the classical variance–covariance inequality which is used in Statistics. Now, we are able to prove Buzano’s inequality which is given by.

Theorem 2.7. For any \(a, b, c \in (H, \langle \cdot, \cdot \rangle)\) we have the inequality

\[
|\langle a, c \rangle \langle b, c \rangle| \leq \frac{1}{2} (|a||b| + |a, b|)|c|^2.
\]

Proof. Theorem 2.1 can be rewritten as

\[
\frac{|\langle a, b \rangle|^2}{|a||b|} + \frac{|\langle b, c \rangle|^2}{|b||c|} + \frac{|\langle c, a \rangle|^2}{|c||a|} \leq 1 + \frac{2|\langle a, b \rangle||\langle b, c \rangle||\langle c, a \rangle|}{|a||b||c|},
\]

which is equivalent to

\[
\left( \frac{a}{|a|} \right) \left( \frac{b}{|b|} \right) + \left( \frac{b}{|b|} \right) \left( \frac{c}{|c|} \right) + \left( \frac{c}{|c|} \right) \left( \frac{a}{|a|} \right) \leq 1 + 2 \left( \frac{a}{|a|} \right) \left( \frac{b}{|b|} \right) \left( \frac{c}{|c|} \right) \left( \frac{a}{|a|} \right).
\]

Denote \(u = \frac{a}{|a|}, v = \frac{b}{|b|}\) and \(x = \frac{c}{|c|}\). Our inequality rewrites as

\[
|\langle u, v \rangle|^2 + |\langle x, u \rangle|^2 + |\langle x, v \rangle|^2 \leq 1 + 2|\langle u, v \rangle|\langle x, u \rangle|\langle x, v \rangle|,
\]

which is equivalent further with

\[
|\langle x, u \rangle|^2 + |\langle x, v \rangle|^2 \leq 1 + |\langle u, v \rangle| - |\langle u, v \rangle| (1 + |\langle u, v \rangle| - 2|\langle u, v \rangle| |\langle x, v \rangle|).
\]

By the arithmetic–geometric mean, we obtain
and we have
\[ 2|\langle x, u \rangle| |\langle x, v \rangle| \leq 1 + |\langle u, v \rangle| - |\langle u, v \rangle||1 + |\langle u, v \rangle| - 2|\langle x, u \rangle| |\langle x, v \rangle||, \]
or
\[ (1 - |\langle u, v \rangle|)(1 + |\langle u, v \rangle| - 2|\langle x, u \rangle| |\langle x, v \rangle||) \geq 0. \]

Then either \(|\langle u, v \rangle| = 1\) which by Cauchy–Schwarz’s inequality, we have \(2 = 1 + |\langle u, v \rangle| \geq 2|\langle x, u \rangle| |\langle x, v \rangle|\) or \(1 > |\langle u, v \rangle|\)
which implies
\[ 1 + |\langle u, v \rangle| - 2|\langle x, u \rangle| |\langle x, v \rangle|| \geq 0, \]
or
\[ |\langle x, u \rangle| |\langle x, v \rangle|| \leq \frac{1}{2}(1 + |\langle u, v \rangle|) \]
which is exactly what we needed to prove. □

**Remark.** In fact, one can prove even more:
\[ |\langle x, u \rangle|^2 + |\langle x, v \rangle|^2 \leq 1 + |\langle u, v \rangle|. \]

Indeed, at some point in the proof of Theorem 2.2 we derived the following inequality
\[ |\langle x, u \rangle|^2 + |\langle x, v \rangle|^2 \leq 1 + |\langle u, v \rangle| - |\langle u, v \rangle||(1 + |\langle u, v \rangle| - 2|\langle x, u \rangle| |\langle x, v \rangle||). \]

By Theorem 2.2 we have
\[ 1 + |\langle u, v \rangle| - |\langle u, v \rangle||(1 + |\langle u, v \rangle| - 2|\langle x, u \rangle| |\langle x, v \rangle||) \leq 1 + |\langle u, v \rangle|. \]

Putting \(u = \frac{a}{\|a\|}, v = \frac{b}{\|b\|}\) and \(x = \frac{c}{\|c\|}\) we obtain the following inequality
\[ (|b||c|a)^2 + (|a||c|b)^2 \leq |a||b|(|a||b| + |a, b|)|c|^2 \]
for all \(a, b, c \in H\).

Other proofs and applications of this last inequality are given in the section that will follow and extensions and generalizations to different settings will be given elsewhere.

## 3. Main result

Now, we are ready to state and prove the main result of this paper. But first, the following result will be crucial in proving our refinement of Buzano’s inequality.

**Theorem 3.1.** Given vectors \(u, v \in H\), with \(|\langle u \rangle||\langle v \rangle| = 1\) (thus \(u, v \in S^{n-1}\)), then
\[
\sup_{|x|=1} (|\langle x, u \rangle|^2 + |\langle x, v \rangle|^2) = 1 + |\langle u, v \rangle|. \]

**First proof.** (Trigonometric proof) Consider the unique vectorial representation \(x = x^* + x^*\) with \(x^* \in \text{span}(u, v)\) and \(x^* \perp \text{span}(u, v)\). Then \(1 = |\langle x \rangle|^2 = (x^* + x^*)^2 = |x^*|^2 + 2(x^*, x^*) = |x^*|^2 + |x^*|^2\) (Pythagoras’ relation), so \(|x^*| < 1\). Now, \(|\langle u, x \rangle| = |\langle u, x^* \rangle| \leq |\langle u, y \rangle|\) where \(y = 0\) if \(x^* = 0\) and \(y = \frac{x^*}{\|x^*\|}\) if \(x^* \neq 0\) (thus \(|\langle x, y \rangle| = 1\)). Similarly \(|\langle v, x \rangle| = |\langle v, x^* \rangle| \leq |\langle v, y \rangle|\). The maximum asked is therefore obtained when \(x \in \text{span}(u, v)\).

Thus our vectorial problem has been transferred in the 2-dimensional space, with unit vectors \(u, v\) and \(x\). The endpoints of the vectors \(\pm u\) and \(y\) partition the circle \(S^1\) into four arcs, each of measure at most \(\pi\) (and possibly 0, when \(u = \pm ve\); the end-point of \(x\) will fall into one of them. Let \(\omega\) be the measure of that arc, and \(\beta\), with \(\alpha + \beta = \omega\), the measures of the arcs between the endpoint of \(x\) and the ends of that arc. Then \(|\langle u, x \rangle|^2 + |\langle v, x \rangle|^2 = \cos^2 \alpha + \cos^2 \beta\), by the well-known geometric interpretation of the dot-product (indeed, of the dimension of the space of \(x\)). But then \(|\langle u, x \rangle|^2 + |\langle v, x \rangle|^2 = \cos^2 \alpha + \cos^2 \beta = 1 + \frac{1}{2}(\cos 2\alpha + \cos 2\beta) = 1 + \cos(\alpha + \beta) \cos(\alpha - \beta) \leq 1 + |\langle x, y \rangle|\), with equality in the obvious cases:

- when \(\alpha = \beta = \omega/2\), therefore when \(|\langle u, x \rangle| = |\langle v, x \rangle|\), therefore \(x = (\pm u \pm v)/||u \pm v||\), where the signs are such that \(0 < \omega < \pi/2\);
- when \(\omega = \pi/2\), therefore when \(|\langle u, v \rangle| = 0\), therefore \(x = (\pm u \pm v)/||u \pm v||\), where the signs are such that \(0 \leq \omega < \pi/2\).

**Second proof.** (Quadratic forms) For \(|\langle x \rangle| = ||u|| = ||v|| = 1\) we have
\[ 0 \leq ||\langle x + \mu u + v v \rangle||^2 = (\langle x + \mu u + v v, x + \mu u + v v \rangle) = \lambda^2 + \mu^2 + \nu^2 + 2\lambda\mu \langle x, u \rangle + 2\lambda v \langle x, v \rangle + 2\mu v \langle u, v \rangle, \]

and
\[ \lambda^2 + \mu^2 + \nu^2 + 2\lambda\mu \langle x, u \rangle + 2\lambda \nu \langle x, v \rangle + 2\nu \mu \langle u, v \rangle \leq 0. \]
a quadratic form which takes non-negative values for any real parameters \( \lambda, \mu, \nu, \) thus corresponding to a positive semidefinite matrix

\[
\begin{bmatrix}
1 & |\langle x, u \rangle| & |\langle x, v \rangle| \\
|\langle x, u \rangle| & 1 & |\langle u, v \rangle| \\
|\langle x, v \rangle| & |\langle u, v \rangle| & 1
\end{bmatrix}
\]

The principal minors of order 1 are thus non-negative, which yields the semi-positivity of the norm; the principal minors of order 2 are non-negative, which yields the Cauchy–Schwarz’s inequality, e.g., \( 1 \geq |\langle u, v \rangle|^2 \); finally, also the determinant of the matrix is non-negative

\[
\Delta = 1 - (|\langle u, v \rangle|^2 + |\langle x, u \rangle|^2 + |\langle x, v \rangle|^2) + 2|\langle u, v \rangle|\langle x, u \rangle\langle x, v \rangle \geq 0,
\]

which can be arranged as \( |\langle x, u \rangle|^2 + |\langle x, v \rangle|^2 \leq 1 + |\langle u, v \rangle| - |\langle u, v \rangle|(|\langle x, u \rangle| - 2|\langle u, v \rangle||\langle x, u \rangle\langle x, v \rangle|) \). But \( |\langle x, u \rangle|^2 + |\langle x, v \rangle|^2 \geq 2|\langle x, u \rangle\langle x, v \rangle| \), which plugged into the relation yields that \( (1 - |\langle u, v \rangle|)(1 + |\langle u, v \rangle| - 2|\langle u, v \rangle||\langle x, u \rangle\langle x, v \rangle|) \geq 0 \). Then either \( 1 - |\langle u, v \rangle| \), when \( 1 + |\langle u, v \rangle| = 2 \geq 2|\langle x, u \rangle| \langle x, v \rangle| \) (by Cauchy–Schwarz), or \( 1 > |\langle u, v \rangle| \), which implies \( 1 + |\langle u, v \rangle| > 2|\langle x, u \rangle\langle x, v \rangle| \). Thus always \( |\langle x, u \rangle|^2 + |\langle x, v \rangle|^2 \leq 1 + |\langle u, v \rangle| \).

Third proof. (Lagrange multipliers) Define

\[
L(x, \lambda) = \langle x, u \rangle^2 + \langle x, v \rangle^2 - \lambda(1 - |x|^2 - 1)
\]

and consider the system

\[
\frac{\partial L}{\partial x_i} = 2u_i\langle x, u \rangle + 2v_i\langle x, v \rangle - 2\lambda x_i = 0
\]

for \( 1 \leq i \leq n \), and

\[
\frac{\partial L}{\partial \lambda} = |x|^2 - 1 = 0.
\]

Now

\[
0 = 1 - \sum_{i=1}^{n} x_i \frac{\partial L}{\partial x_i} = 2u_i\langle x, u \rangle + 2v_i\langle x, v \rangle - \lambda - \sum_{i=1}^{n} x_i^2 = \langle x, u \rangle^2 + \langle x, v \rangle^2 - \lambda|\langle x, u \rangle|\langle x, v \rangle| = \langle x, u \rangle^2 + \langle x, v \rangle^2 - \lambda,
\]

so \( \lambda \) needs be, computed at the critical point(s), the very expression we are considering (of no relevance, in fact). On the other hand,

\[
0 = 1 - \sum_{i=1}^{n} u_i \frac{\partial L}{\partial x_i} = \langle x, u \rangle\sum_{i=1}^{n} u_i^2 + \langle x, v \rangle\sum_{i=1}^{n} v_i - \lambda - \sum_{i=1}^{n} x_i^2 u_i = \langle x, u \rangle\|u\|^2 + \langle x, v \rangle\langle u, v \rangle - \lambda\langle x, u \rangle
\]

and similarly for \( v \), thus reaching the system of two equations (in the variables \( \langle x, u \rangle \) and \( \langle x, v \rangle \))

\[
\begin{cases}
(1 - \lambda)\langle u, v \rangle + \langle u, v \rangle\langle x, v \rangle = 0, \\
\langle x, v \rangle\langle u, v \rangle + (1 - \lambda)\langle x, v \rangle = 0.
\end{cases}
\]

The determinant \( \Delta \) of the matrix of this particular system is \( \Delta = (1 - \lambda)^2 - \langle u, v \rangle^2 \), and, if not null, the only solution is the trivial \( \langle x, u \rangle = \langle x, v \rangle = 0 \), when our expression reaches an evident minimum, equal to zero (therefore since then \( \lambda = 0 \), this means \( \langle u, v \rangle \neq \pm 1 \), which translates in \( u \neq \pm v \), and \( x \perp \text{span}(u, v) \)). We are therefore interested in \( \Delta = 0 \) (for all other critical points), leading to \( \lambda = 1 \pm |\langle u, v \rangle| \) at a maximum point, and \( \lambda = 1 - |\langle u, v \rangle| \) at a critical point. There are some particular cases.

When \( u = \pm v \), thus \( \langle u, v \rangle = \pm 1 \), the situation is simple. We have a maximum of 2 when \( x = \pm u \), and a minimum of 0 when \( x \perp u \).

When \( u \perp v \), thus \( \langle u, v \rangle = 0 \), then \( \lambda \) (thus the expression) is 1 at the maximum points, for all \( x \in \text{span}(u, v) \), and \( \lambda \) (thus the expression) is 0 at the minimum points, for all \( x \perp \text{span}(u, v) \). \( \square \).

The main result of this paper is given by

**Theorem 3.2.** Let \( (H, \langle \cdot, \cdot \rangle) \) be a real or complex inner product space and \( a, b, c \in H \). Then

\[
(||a||b, c)|^2 + (||b||a, c)|^2 \leq ||a||b||a||b|| + (a, b)||c||^2.
\]

**Proof.** For any not-null \( a, b, c \), let us take \( u = \frac{a}{||a||} \), \( v = \frac{b}{||b||} \), \( x = \frac{c}{||c||} \) in Theorem 2.1 and, therefore \( \langle x, u \rangle^2 = \frac{1}{||a||^2}||a||^2 \), \( \langle x, v \rangle^2 = \frac{1}{||b||^2}||b||^2 \), \( \langle u, v \rangle = \frac{a b}{||a|| ||b||} \), so the proven relation becomes

\[
\langle x, u \rangle^2 + \langle x, v \rangle^2 \leq ||a||b||a||b|| + (a, b)||c||^2.
\]
\[
\frac{1}{|a|} |(a, c)|^2 + \frac{1}{|b|} |(b, c)|^2 \leq \left( 1 + \frac{|(a, b)|}{|a| - |b|} \right) |c|^2 ,
\]
equivalent with the required inequality, also true for \(a, b\) or \(c\) equal to zero. \(\square\)

**Remark.** We can continue by Arithmetic–Geometric mean to obtain
\[
|(u, x) \cdot (v, x)| \leq \frac{1}{2} \left( |(u, x)|^2 + |(v, x)|^2 \right) \leq \frac{1}{2} \left( 1 + |(u, v)| \right).
\]
A simple computation now yields
\[
|(a, c) \cdot (b, c)| \leq \frac{1}{2} (|a| - |b| + |(a, b)|) |c|^2 ,
\]
also true for \(a, b\) or \(c\) equal to zero, which is exactly Buzano's inequality.

As a consequence of Theorem 3.2 we have the general integral version.

**Corollary 3.3.** Let \((H, \Sigma, \mu)\) be a positive measure space and let \(f, g, h \in L^2(H, \Sigma, \mu)\) the Hilbert space of complex-valued \(2 - \mu\) integrable functions defined on \(H\). Then
\[
\left( \int_H f^2(t)d\mu(t) \right) \left( \int_H g(t)h(t)d\mu(t) \right)^2 \leq \left( \int_H f^2(t)d\mu(t) \right) \left( \int_H g^2(t)d\mu(t) \right) \left( \int_H h(t)f(t)d\mu(t) \right)^2 \leq \left( \int_H f^2(t)d\mu(t) \right)^{1/2} \left( \int_H g^2(t)d\mu(t) \right)^{1/2} \left( \int_H h^2(t)d\mu(t) \right)^{1/2}.
\]

It might be useful to see that out of Theorem 3.2 one can get the following discrete version.

**Corollary 3.4.** If \(p_i \geq 0, i = 1, 2, \ldots, n\) and \(a_i, b_i, c_i\) are complex numbers, then
\[
\left( \sum_{i=1}^n p_i |a_i|^2 \right) \left( \sum_{i=1}^n p_i |b_i|^2 \right)^{1/2} \leq \left( \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i |b_i|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i |c_i|^2 \right)^{1/2}.
\]

In the case when \(H\) is the space of real-valued continuous functions defined on the interval \([a, b]\) endowed with the inner product \(<f, g> := \int_a^b f(x)g(x)dx\), we have.

**Corollary 3.5.** Let \(f, g, h\) be continuous real-valued functions on \([a, b]\). Then, we have the following inequality
\[
\left( \int_a^b f^2(t)dt \right) \left( \int_a^b g(t)h(t)dt \right)^2 \leq \left( \int_a^b f^2(t)dt \right) \left( \int_a^b g^2(t)dt \right) \left( \int_a^b h(t)f(t)dt \right)^2 \leq \left( \int_a^b f^2(t)dt \right)^{1/2} \left( \int_a^b g^2(t)dt \right)^{1/2} \left( \int_a^b h^2(t)dt \right)^{1/2} \left( \int_a^b f^2(t)dt \right)^{1/2} \left( \int_a^b g^2(t)dt \right)^{1/2} \left( \int_a^b h^2(t)dt \right)^{1/2}.
\]
The discrete real version of Corollary 2.4 is given by.

**Corollary 3.6.** Given real numbers \(a_i, b_i, c_i, 1 \leq i \leq n\), then
\[
\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \left( \sum_{i=1}^n c_i^2 \right)^{1/2}.
\]

### 4. Some elementary applications

In [16] Wang and Zhang proved an inequality for unitary matrices. Now, we are able to give other trace inequalities for unitary matrices. A similar inequality was obtained by Zhang [17] but with a slightly different method.

**Theorem 4.1.** For a complex \(n \times n\) matrix \(X\) denote \(m(X)\) the algebraic mean of its eigenvalues. For any two unitary \(n \times n\) complex matrices \(U, V\), we have
\[ |m(U)|^2 + |m(V)|^2 + |m(UV)|^2 \leq 1 + 2|m(U)||m(V)||m(UV)| \]

and
\[ |m(U)|^2 + |m(V)|^2 \leq 1 + |m(UV)|. \]

**Proof.** Let \( c = \frac{1}{\sqrt{n}}, \ a = \frac{1}{\sqrt{n}} \ U \) and \( b = \frac{1}{\sqrt{n}} \ V \) where \( U \) and \( V \) are unitary matrices. Clearly, \( ||a|| = ||b|| = ||c|| = 1 \) and moreover
\[
\langle a, b \rangle = \frac{1}{n} \text{tr}(UV^*) = \frac{1}{n} \text{tr}(UV),
\]
\[
\langle b, c \rangle = \frac{1}{n} \text{tr}(U), \quad \langle c, a \rangle = \frac{1}{n} \text{tr}(V).
\]

Now, by Theorems 2.1 and 3.2 we have
\[
\frac{1}{n^2}(|\text{tr}(U)|^2 + |\text{tr}(V)|^2 + |\text{tr}(UV)|^2) \leq 1 + \frac{2}{n^3}(|\text{tr}(U)||\text{tr}(V)||\text{tr}(UV)|)
\]
and
\[
\left( \frac{1}{n} |\text{tr}(U)| \right)^2 + \left( \frac{1}{n} |\text{tr}(V)| \right)^2 \leq 1 + \frac{1}{n} |\text{tr}(UV)|.
\]

Taking into account that \( m(X) = \frac{1}{n} \text{tr}(X) \), we obtain exactly our inequalities. \( \square \)

**Remark.** In general, for the Hilbert–Schmidt inner product, \( \langle X, Y \rangle := \text{tr}(X^* Y) \), for any complex square matrices \( A, B \) and \( C \), we have the following inequalities
\[
\text{tr}(AA^*)|\text{tr}(BC^*)|^2 + |\text{tr}(BB^*)||\text{tr}(CA^*)|^2 + |\text{tr}(CC^*)||\text{tr}(CA^*)|^2 \leq \text{tr}(AA^*)|\text{tr}(BC^*)| \text{tr}(CC^* + 2 |\text{tr}(AB^*)| \text{tr}(BC^*)| \text{tr}(CA^*)|
\]
and
\[
\text{tr}(AA^*) \cdot |\text{tr}(BB^*)|^2 + |\text{tr}(BB^*)| \cdot |\text{tr}(CA^*)|^2 \leq \text{tr}^{1/2}(AA^*) \cdot \text{tr}^{1/2}(BB^*) |\text{tr}(CC^*)| \text{tr}^{1/2}( AA^*) \cdot \text{tr}^{1/2}(BB^*) + |\text{tr}(A'B)|.
\]

**Theorem 4.2.** For reals \( a_i, b_i, c_i, i = 1, 2, \ldots, n \) such that \( \sum_{i=1}^n a_i b_i = 0 \), we have
\[
\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \geq 4 \left( \sum_{i=1}^n a_i c_i \right) \left( \sum_{i=1}^n b_i c_i \right)^2
\]
and
\[
\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) + \left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \geq 4 \left( \sum_{i=1}^n a_i c_i \right) \left( \sum_{i=1}^n b_i c_i \right)^2.
\]

**Proof.** By Arithmetic–Geometric mean inequality, we have
\[
\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i c_i \right)^2 + \left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) \geq 2 \left( \sum_{i=1}^n b_i c_i \right) \left( \sum_{i=1}^n c_i a_i \right) \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2}.
\]
By Corollary 3.6 and taking into account that \( \sum_{i=1}^n a_i b_i = 0 \), we obtain
\[
\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) \geq 2 \left( \sum_{i=1}^n b_i c_i \right) \left( \sum_{i=1}^n c_i a_i \right) \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2},
\]
which is equivalent with the desired inequality. As for the second inequality, applying again arithmetic–geometric mean inequality, we have
\[
\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) \left( \sum_{i=1}^n a_i c_i \right)^2 + \left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) \left( \sum_{i=1}^n b_i c_i \right)^2 \geq 2 \sqrt{\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \left( \sum_{i=1}^n c_i^2 \right) \left( \sum_{i=1}^n a_i c_i \right) \left( \sum_{i=1}^n b_i c_i \right)}
\]
and by using the first inequality we finally obtain the second one. \( \square \)
\textbf{Theorem 4.4.} If \(m, n\) are positive integers and \(A, B\) and \(C\) are \(m \times n\) real matrices, then
\[
\det(A^T) \det(B^T) + \det(B^T) \det(C^T) + \det(B^T) \det(C^T) + \det(A^T) \det(B^T) \det(C^T) 
\leq \det(A^T) \det(B^T) \det(C^T)
\]
and
\[
\det(A^T) \det(B^T) \det(C^T) + \det(B^T) \det(C^T) + \det(A^T) \det(B^T) \det(C^T) 
\leq \det(A^T) \det(B^T) \det(C^T)
\]

\textbf{Proof.} If \(m > n\) then \(A^T, B^T, C^T\) and \(AA^T, BB^T, CC^T\) are \(m \times m\) matrices with rank at most \(m\), so our inequality is obvious. So, we may assume without loss of generality that \(m \leq n\). Consider \(S = \{j_1, j_2, \ldots, j_m : 1 \leq j_1 < \cdots < j_m \leq n\}\). Now, for a \(m \times n\) matrix \(X\) with real entries and \(w = (w_1, w_2, \ldots, w_m) \in S\) denote by \(X_w\) the \(m \times m\) submatrix whose columns are columns \(j_1, j_2, \ldots, j_m\) of \(X\) for \(X, Y\) two \(m \times n\) matrices with real entries, by the Cauchy-Binet formula, we have
\[
\det(XY^T) = \sum_{w \in S} \det X_w \det Y_w.
\]
with respect to the inner product \((\cdot, \cdot) : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}\). Now, if \(a = \det(A_w), b = \det(B_w)\) and \(c = \det(C_w)\), then by applying \textit{Theorems 2.1 and 3.2}, we obtain
Theorem 4.5. For \( n \geq 2 \) positive integer, \( a, b \), \( i = 1, 2, \ldots, n \), such that \( \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1, \sum_{i=1}^{n} a_i b_i = 0 \), we have
\[
\left( \sum_{i=1}^{n} a_i \right)^2 + \left( \sum_{i=1}^{n} b_i \right)^2 \leq n.
\]

**Proof.** Take \( c_i = 1, i = 1, 2, \ldots, n \) in Corollary 3.6. \( \square \)

The integral analog of Theorem 4.2 is given by.

Theorem 4.6. Let \( f, g, h \) be continuous real-valued functions on \([0, 1]\) satisfying the condition \( \int_{[0,1]} fg = 0 \). Then, we have
\[
\left( \int_{[0,1]} f^2 \right) \left( \int_{[0,1]} g^2 \right) \left( \int_{[0,1]} h^2 \right) \geq 4 \left( \int_{[0,1]} fh \right)^2 \left( \int_{[0,1]} gh \right)^2
\]
and
\[
\left( \int_{[0,1]} f^2 \right) \left( \int_{[0,1]} g^2 \right) \left( \int_{[0,1]} h^2 \right) \geq 4 \left( \int_{[0,1]} fh \right)^2 \left( \int_{[0,1]} gh \right)^2.
\]

**Proof.** Let us consider by \( C([0,1]) \) be the space of continuous functions endowed with the inner product
\[
\langle f, g \rangle := \int_{[0,1]} f(x)g(x)dx
\]
and the associated norm \( \|f\| = \left( \int_{[0,1]} f^2(x)dx \right)^{1/2} \). Our inequalities can be rewritten as
\[
\|f\|^2 \|h\|^2 \cdot \|g\|^2 \|h\|^2 \geq 4 \|gh\|^2 \cdot \|hf\|^2, \tag{5}
\]
and
\[
\|f\|^2 \|h\|^2 \langle f, h \rangle^2 + \|g\|^2 \|h\|^2 \langle g, h \rangle^2 \geq 4 \langle g, h \rangle^2 \cdot \langle h, f \rangle^2, \tag{6}
\]
by the Corollary 3.4, we have
\[
(||f||g, h)||^2 + (||g||f, h)||^2 \leq ||f||g|| ||f||g|| + ||f, g)||^2 ||h||^2
\]
for any \( f, g, h \in C([0,1]) \). On the other hand, by Arithmetic–Geometric mean inequality we obtain
\[
(||f||g, h)||^2 + (||g||f, h)||^2 \geq 2||f||g|| ||g, h)||^2 ||h||^2.
\]
Combining this inequality with the one obtained from the Corollary 2.5 and taking into account that \( f \cdot g = 0 \), we get
\[
|f|^2 |g|^2 |h|^2 \geq 2 |f||g| \cdot |f, h||h, f|,
\]
which is finally equivalent to
\[
|f|^2 |g|^2 |h|^2 \geq 4 |g, h|^2 |h, f|^2
\]
and thus the first inequality is proved. For the second inequality, we shall use the first inequality. Hence, again by Arithmetic–Geometric mean inequality we have
\[
|f|^2 |g|^2 |h|^2 = |f||g| \cdot |f| |g, h| |h, f| \cdot |h, f|^{1/2} |h, f|^{1/2} \geq 2 |f||g| \cdot |g, h| \cdot |h, f| \cdot |h, f| \cdot |h, f|^{1/2} \cdot |h, f|^{1/2} = 2 |g, h|^2 |h, f|^2.
\]
Now, we only need to see that
\[
2 |f||g| |h||h, f| \cdot |g, h| \cdot |g, h| \geq 4 |g, h|^2 |h, f|^2.
\]
This last inequality finally reduces to
\[
|f||g||h|^2 \geq 2 |g, h||h, f|
\]
and this exactly inequality (5) written in the form
\[
|f|^2 |g|^2 |h|^2 \geq 4 |g, h|^2 |h, f|^2. \quad \Box
\]

Remark. For \( h = 1 \), we obtain [10].

**Theorem 4.7.** ([11]) Let \( n \) be a positive integer, and write a vector \( x = (x_1, x_2, \ldots, x_n) \). For \( x, y, a, b \in \mathbb{R}^n \) let
\[
[x, y]_{a,b} = \sum_{1 \leq i < j \leq n} x_i y_j \min(a_i, b_j).
\]

Then, for \( x, y, z, a, b, c \in \mathbb{R}^n \) with nonnegative entries,
\[
[x, x]_{a,a} \cdot [y, z]_{b,b} + [y, y]_{b,b} \cdot [z, z]_{c,c} \cdot [x, y]_{a,b}^2 \leq [x, x]_{a,a} \cdot [y, z]_{b,b} \cdot [z, z]_{c,c} + 2 [x, y]_{a,b} \cdot [y, ]_{b,b} \cdot [x, x]_{a,a}.
\]

and
\[
[x, x]_{a,a} \cdot [y, z]_{b,b}^2 + [y, y]_{b,b} \cdot [z, z]_{c,c} \cdot [x, y]_{a,b} \leq [x, x]_{a,a} \cdot [y, y]_{b,b} \cdot [z, z]_{c,c} \cdot \left( [x, x]_{1/2}^1 \cdot [y, y]_{1/2}^1 + [x, y]_{1/2} \right).
\]

**Proof.** Our inequalities can be restated as
\[
\left( \sum_{1 \leq i < j \leq n} x_i y_j \min(a_i, a_j) \right) \cdot \left( \sum_{1 \leq i < j \leq n} y_i z_j \min(b_j, b_j) \right) + \left( \sum_{1 \leq i < j \leq n} z_i y_j \min(c_j, c_j) \right) \leq \left( \sum_{1 \leq i < j \leq n} x_i y_j \min(a_i, b_j) \right) \cdot \left( \sum_{1 \leq i < j \leq n} y_i z_j \min(b_j, c_j) \right) \cdot \left( \sum_{1 \leq i < j \leq n} z_i y_j \min(c_j, b_j) \right)
\]
and
\[
\left( \sum_{1 \leq i < j \leq n} x_i y_j \min(a_i, a_j) \right) \cdot \left( \sum_{1 \leq i < j \leq n} y_i z_j \min(b_j, c_j) \right) \leq \left( \sum_{1 \leq i < j \leq n} x_i y_j \min(b_j, b_j) \right) \cdot \left( \sum_{1 \leq i < j \leq n} z_i y_j \min(c_j, c_j) \right) \cdot \left( \sum_{1 \leq i < j \leq n} z_i y_j \min(c_j, a_j) \right)
\]

Let \( \chi_A \) be the characteristic function of an arbitrary set \( A \). Consider the functions \( f, g, h : [0, \infty) \to \mathbb{R} \) defined by
\[
f(x) = \sum_{i=1}^{n} \chi_{[0,a_i]}(x),
\]
\[ g(x) = \sum_{i=1}^{n} y_i Z_{0:b_i}(x). \]

\[ h(x) = \sum_{i=1}^{n} z_i \chi_{[0,c_i]}(x). \]

We have
\[
\int_{0}^{\infty} f^2(x)dx = \sum_{1 \leq i < j \leq n} x_1 x_2 \int_{0}^{\infty} Z_{i:0:a_i}(x) Z_{j:0:a_j}(x)dx = \sum_{1 \leq i < j \leq n} x_1 x_2 \min(a_i, a_j).
\]

Analogously, we obtain
\[
\int_{0}^{\infty} g^2(x)dx = \sum_{1 \leq i < j \leq n} y_1 y_2 \min(b_i, b_j), \quad \int_{0}^{\infty} h^2(x)dx = \sum_{1 \leq i < j \leq n} z_1 z_2 \min(c_i, c_j),
\]
as well as
\[
\int_{0}^{\infty} f(x)g(x)dx = \sum_{1 \leq i < j \leq n} x_1 y_2 \min(a_i, b_j),
\]

\[
\int_{0}^{\infty} g(x)h(x)dx = \sum_{1 \leq i < j \leq n} y_1 z_2 \min(b_i, c_j)
\]
and
\[
\int_{0}^{\infty} g(x)h(x)dx = \sum_{1 \leq i < j \leq n} z_1 x_2 \min(c_i, a_j).
\]

Finally, by applying Theorem 2.1 and Corollary 3.3 with the inner product given by \( (f, g) := \int_{-\infty}^{\infty} f(x)g(x)dx \), with \( f, g \) defined as above, we derive the inequalities
\[
\int_{0}^{\infty} f^2(x)dx \left( \int_{0}^{\infty} g(x)h(x)dx \right)^2 + \int_{0}^{\infty} g^2(x)dx \left( \int_{0}^{\infty} h(x)f(x)dx \right)^2 + \int_{0}^{\infty} h^2(x)dx \left( \int_{0}^{\infty} f(x)g(x)dx \right)^2
\]
\[
\leq \left( \int_{0}^{\infty} f^2(x)dx \right)^{1/2} \left( \int_{0}^{\infty} g^2(x)dx \right)^{1/2} \left( \int_{0}^{\infty} h^2(x)dx \right)^{1/2} \left( \int_{0}^{\infty} f^2(x)dx \right)^{1/2} \left( \int_{0}^{\infty} g^2(x)dx \right)^{1/2} + \int_{0}^{\infty} f(x)g(x)dx
\]
and, now the conclusion is obvious. \( \Box \)

**Remark.** These kind of min–max inequalities for vectors arise in coding of messages. A list of such inequalities is provided in [3].

**Theorem 4.8.** For a matrix \( A \in M_n(\mathbb{R}) \), let \( f_A : M_n(\mathbb{R}) \to \mathbb{R} \)
\[ f_A(X) = \sum_{i,j=1}^{n} b_{ij} x_{ij}, \]
where \( B = [b_{ij}]_{1 \leq i,j \leq n} = AA^T \). Then, we have
\[ f_A(XX^T) \cdot f_A(YZ^T) + f_A(YY^T) \cdot f_A(ZZ^T) \leq f_A(XX^T) \cdot f_A(YY^T) + 2 f_A(YY^T) \cdot f_A(ZZ^T) \cdot f_A(XZ^T) \]
and
\[ f_A(XX^T) \cdot f_A(YZ^T) + f_A(YY^T) \cdot f_A(ZX^T) \leq f_A^{1/2}(XX^T) \cdot f_A^{1/2}(YY^T) \cdot f_A(ZZ^T) \left( f_A^{1/2}(XX^T) \cdot f_A^{1/2}(YY^T) + f_A(XY^T) \right). \]
Proof. We define the inner product

\[ \langle X, Y \rangle := f_A(XY^T) = \sum_{1 \leq i, j \leq n} b_{ij}x_i y_j = \sum_{k=1}^n \text{column}_k(X)^T B \text{column}_k(Y). \]

Moreover, since \( B \) is positive semidefinite, we have

\[ f_A(XX^T) = \sum_{k=1}^n \text{column}_k(X)^T B \text{column}_k(X) \]

and clearly, \( f_A(XX^T) = 0 \) implies that \( \text{column}_k(X) = 0 \) for every \( k \), that is \( X = 0 \). Now, by Theorems 2.1 and 3.2, we derive our inequalities. \( \square \)

Theorem 4.9. Let \( A = [a_{ij}]_{1 \leq i, j \leq n} \in M_n(\mathbb{R}) \), be a symmetric matrix with \( a_{ii} > \sum_{j=1}^n |a_{ij}| \) and \( x_i, y_j, i = 1, 2, \ldots, n \) are nonnegative real numbers. Then, we have

\[ \left( \sum_{1 \leq i, j \leq n} a_{ij}x_i y_j \right)^2 \leq \left( \sum_{1 \leq i, j \leq n} a_{ij}x_i \right)^2 \left( \sum_{1 \leq i, j \leq n} a_{ij}y_j \right)^2. \]

Proof. For vectors \( x, y \in \mathbb{R}^n \), define the function \( f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \)

\[ f(x, y) := \sum_{1 \leq i, j \leq n} a_{ij}x_i y_j. \]

Clearly, \( f(x, y) = x \cdot A y \).

The only property that we need to verify is \( f(x, x) > 0 \) for \( x \neq 0 \). We have

\[ f(x, x) = \sum_{1 \leq i, j \leq n} a_{ij}x_i y_j = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij}x_i y_j \right) = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij}x_j \right) y_i = \sum_{i=1}^n |a_{ij}|(x_i^2 + y_i^2) + 2 \sum_{i<j} a_{ij}x_i y_j \]

\[ \geq 2 \sum_{i=1}^n |a_{ij}| |x_i| |x_j| + 2 \sum_{i<j} a_{ij}x_i y_j \geq 0. \]

Thus, \( f(x, y) \) is an inner product on \( \mathbb{R}^n \). Now, by applying Theorems 2.1 and 3.2, we obtain our result. \( \square \)

Remark. More generally, one can prove that if \( A \) is a positive definite symmetric matrix with real entries, then the function \( f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \)

\[ f(x, y) = \sum_{1 \leq i, j \leq n} a_{ij}x_i y_j, \]

defines an inner product on \( \mathbb{R}^n \). Thus, Theorem 4.8 holds in a slightly general setting.

One can consider some classical examples of positive definite and symmetric matrices. For instance, the Hilbert matrix

\[ A_1 = (a_{ij})_{1 \leq i, j \leq n} = \begin{bmatrix} 1 & \frac{1}{1^2} & \cdots & \frac{1}{1^n} \\ \frac{1}{2^2} & 1 & \cdots & \frac{1}{2^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n^2} & \frac{1}{(n-1)^2} & \cdots & 1 \end{bmatrix} \]

which is positive definite. This is equivalent with showing that for positive \( x_i, i = 1, 2, \ldots, n \) we have \( \sum_{1 \leq i, j \leq n} x_i x_j > 0 \). For instance, one ca easily prove that

\[ \sum_{1 \leq i, j \leq n} \frac{x_i x_j}{i+j-1} = \int_0^1 \left( \sum_{i=1}^n x_i t^{i-1} \right)^2 dt > 0. \]
when $f_1(t) = \left( \sum_{i=1}^{n} x_i d_i^{-1} \right)^2$ is not the zero function. The same for Cauchy matrix, $A_2 = (a_{ij})_{1 \leq i,j \leq n} = \left[ \frac{x_j}{1 + x_i} \right]_{1 \leq i,j \leq n}$, one can prove that

$$
\sum_{1 \leq i,j \leq n} X_i X_j = \int_0^{\infty} \left( \sum_{k=1}^{n} x_k e^{-x} \right)^2 dx > 0.
$$

when $f_2(t) = \left( \sum_{i=1}^{n} x_i e^{-x} \right)^2$ is not the zero function.

On the other hand, on the axis of real numbers we consider the intervals $I_i$, $i = 1, 2, \ldots, n$ and denote by $a_{ij}$ be the length of the interval $I_i \cap I_j$, for $i = 1, 2, \ldots, n$. One can prove that the matrix $A_3 = (a_{ij})_{1 \leq i,j \leq n} = \left[ \text{length}(I_i \cap I_j) \right]_{1 \leq i,j \leq n}$ is positive definite. Indeed, denote $I = \bigcup_{i=1}^{n} I_i$ and for $J \subset I$ consider the characteristic function of the set $J$, $X_J : I \rightarrow \{0,1\}$. If $J$ is an interval, then the length of $J$ is given by $\text{length}(J) = \int_J X_J(x) dx = \int_J X_J^2 dx$. On the other hand,

$$
a_{ij} = \int X_i(x) X_j(x) dx
$$

and thus,

$$
\sum_{1 \leq i,j \leq n} a_{ij} x_i x_j = \int \left( \sum_{1 \leq i,j \leq n} X_i(x) X_j(x) x_i x_j \right) dx = \int \left( \sum_{i=1}^{n} x_i X_i(x) \right)^2 dx > 0
$$

for positive $x_i$, $i = 1, 2, \ldots, n$.

The same idea works when $A_4 = (a_{ij})_{1 \leq i,j \leq n} = \text{area}(D_i \cap D_j)$, where $D_i$, $i = 1, 2, \ldots, n$ are discs in the Euclidian plane. Indeed, we have the characteristic function of the set $D_i$, $X_{D_i} : \mathbb{R}^2 \rightarrow \{0,1\}$, and moreover

$$
\text{area}(D_i) = \int \int_{\mathbb{R}^2} X_{D_i}(x,y) dxdy = \int \int_{\mathbb{R}^2} X_{D_i}^2(x,y) dxdy
$$

and

$$
\text{area}(D_i \cap D_j) = \int \int_{\mathbb{R}^2} X_{D_i}(x,y) X_{D_j}(x,y) dxdy.
$$

Now, we have

$$
\sum_{1 \leq i,j \leq n} a_{ij} x_i x_j = \int \int_{\mathbb{R}^2} \sum_{1 \leq i,j \leq n} x_i X_{D_i}(x,y) x_j X_{D_j}(x,y) dxdy = \int \int_{\mathbb{R}^2} \left( \sum_{i=1}^{n} x_i X_{D_i}(x,y) \right)^2 dxdy > 0.
$$

Last but not least, we end the note with an elementary inequality with its reminiscents in the theory of improper integrals.

**Theorem 4.10.** For all nonnegative reals $a, b, c$ we have the inequalities

$$
\frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} \leq \frac{1}{4} + \frac{4abc}{(a+b)(b+c)(c+a)}
$$

and

$$
\frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} \leq \frac{1}{8abc} + \frac{1}{4c(a+b)}.
$$

**Proof.** The key is the following identity:

$$
\frac{1}{a+b} = \int_0^\infty e^{-(a+b)x} dx.
$$

Define the inner product $\langle f, g \rangle := \int_0^\infty f(x)g(x) dx$, where $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. A short computation shows that $\int_0^\infty e^{-ax} dx = \frac{1}{a^2}$. So by Theorem 2.1 we have

$$
\frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{1}{(a+b)^2} \leq \frac{1}{8abc} + \frac{2}{(a+b)(b+c)(c+a)},
$$

which is equivalent to the first inequality.
Now, by Corollary 3.3, we obtain
\[
\frac{1}{a(b + c)^2} \cdot \frac{1}{b(c + a)^2} \leq \frac{1}{4abc^2} \left( \frac{1}{2ab} + \frac{1}{a + b} \right),
\]
which is our second inequality. \(\square\)

Acknowledgements

The first author would like to thank Politehnica University of Bucharest for hospitality during May–June 2012 when part of this work was initiated. The authors would also like to thank Beniamin Bogosel for crucial observations which improved the initial version of this paper.

References