The questions are independent: Not answering a question has no impact on the subsequent questions, but you may have to use the result of an unanswered question to answer the next one.

1) Let \((M, d)\) be a metric space.
   (a) Suppose that there is a sequence \(B_n\) of connected subsets of \(M\) such that \(B_n \subset B_{n+1}\) and that \(\bigcup_n B_n = M\). Show that \(M\) is connected (argue by contradiction.)
   (b) Deduce from (a) that if \(B_n\) is a sequence of connected subsets of \(M\) such that \(B_n \subset B_{n+1}\), then \(\bigcup_n B_n\) is connected. (This is a one-liner; the only difference with (a) is that \(\bigcup_n B_n = M\) is not assumed.)
   (c) Let \(A_n\) be a sequence of connected subsets of \(M\) such that, for every \(n\), there is \(j \leq n\) such that \(A_{n+1} \cap A_j \neq \emptyset\). Show that \(\bigcup_n A_n\) is connected.
      (Hint: Find \(B_n\) satisfying the conditions in (b) such that \(\bigcup_n B_n = \bigcup_n A_n\). To do this, recall that if \(C, D\) are connected subsets of \(M\) such that \(C \cap D \neq \emptyset\), then \(C \cup D\) is connected.)
   (d) The assumptions made in (c) imply that for every \(n\), there is \(n \neq n\) such that \(A_n \cap A_n \neq \emptyset\). Find a simple example with each \(A_n\) connected (for instance, intervals in \(R\)) showing that this does not ensure that \(\bigcup_n A_n\) is connected.

2) (a) Let \(f : (a, b) \to R\) have a local maximum (minimum) at \(x_0\). Show that, if \(f\) is twice differentiable, then \(f''(x_0) \leq 0\) (\(f''(x_0) \geq 0\)). (Use Taylor approximation and \(f''(x_0) = 0\).
   (b) Suppose now that \(f : R \to R\) is twice differentiable and that \(x^2 f''(x) + x f'(x) - f(x) = 0\) for all \(x \in R\). Show that \(f\) cannot have a strictly positive maximum or a strictly negative minimum at any point \(x_0 \in R\) (use (a)).
   (c) Deduce from (b) that if also \(\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} f(x) = 0\), then \(f = 0\) (argue by contradiction).

3) Let \(f : [0, \infty) \to R\) be a function such that \(\lim_{x \to \infty} f(x) = a \in R\).
   (a) Show that given \(\varepsilon > 0\), there is \(T > 0\) such that
      \[
      \{x > T \text{ and } y > T\} \Rightarrow |f(x) - f(y)| < \varepsilon.
      \]
(b) With $\varepsilon$ and $T$ as in part (a), show that there is $\delta > 0$ such that
\[ x_0 > T + \delta \text{ and } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \]

(c) Deduce from (b) that $f$ is uniformly continuous on $[0, \infty)$. (Hint: $[0, T + \delta]$ is compact.)

4) Let $(M, d_M)$ and $(N, d_N)$ be metric spaces and let $f : M \rightarrow N$ be a continuous mapping.

(a) Suppose that $f^{-1}(C)$ is compact for every compact subset $C \subset N$. Show that if $E$ is a closed subset of $M$, then $f(E)$ is closed in $N$. (Hint: Recall that in a metric space, the set consisting of a convergent sequence together with its limit is always compact.)

(b) Suppose $M = N = \mathbb{R}$ equipped with the euclidean metric (and $f$ continuous). Show that $f^{-1}(C)$ is compact whenever $C$ is compact if and only if $\lim_{t \to -\infty} |f(t)| = \infty$.

(c) Show that if $M$ is compact and $f$ is continuous, then $f^{-1}(C)$ is compact for every compact $C \subset N$.

(d) Show that if $M$ is not compact, there is a continuous function $f : M \rightarrow N$ such that $f(E)$ is closed for every subset $E$ of $M$ (even if $E$ is not closed), but $f^{-1}(C)$ is not compact for some compact subset $C$ of $N$. (This is actually a one-liner.)