Problem 1. Find the characteristic polynomial, eigenvalues and eigenvectors for each of the following matrices:

a) \( A = \begin{bmatrix} 5 & 4 \\ -8 & -7 \end{bmatrix} \)

b) \( A = \begin{bmatrix} -2 & 5 \\ 0 & 2 \end{bmatrix} \)

c) \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix} \)

d) \( A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{bmatrix} \).

Decide which of the above matrices is invertible and compute its inverse, if the case.

Problem 2. Find the characteristic polynomial of

\( A = \begin{bmatrix} 5 & 4 \\ -8 & -7 \end{bmatrix} \).

and show that \( A^2 + 2A - 3I \) equals the zero matrix.

Problem 3. Solve the following systems:

a) \( y' = \begin{bmatrix} 2 & 0 \\ -4 & -2 \end{bmatrix} y. \)

b) \( y' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} y. \)

c) \( x' = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix} x. \)

d) \( x' = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x. \)
Problem 4. Suppose $A$ is a real $2 \times 2$ matrix with one eigenvalue $\lambda$ of multiplicity 2. Show that the solution to the initial value problem

$$y' = Ay, y(0) = v,$$

is given by

$$y(t) = e^{\lambda t}[v + t(A - \lambda I)v].$$

Problem 5. Solve the following systems:

a) $x' = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} x.$

b) $x' = \begin{bmatrix} -3 & -6 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix} x.$

Problem 6. Find the Fourier series for the following functions:

a) $f(x) = 4 - x^2$ on $[-2, 2]$

b) $f(x) = x^3$ on $[-1, 1]$.

Problem 7. a) Compute the Fourier series for the function $f(x) = |x|$ on the interval $[-\pi, \pi]$.

b) Show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n + 1)^2} = \frac{\pi^2}{8}.$$ 

Problem 8. a) Compute the Fourier series for the function $f(x) = x^2$ on the interval $[-\pi, \pi]$.

b) Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n+1}{n^2} = \frac{\pi^2}{12},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$
Problem 9. a) Compute the Fourier series for the function $f(x) = x^4$ on the interval $[-\pi, \pi]$.

b) Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{120},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Problem 10. Show that

$$\int_{0}^{1} \sin(2n\pi x) \cos(2n\pi x) dx = 0.$$