Math 1540 Advanced Calculus 2
Midterm #1 (Friday, Feb. 20, 2015)

4 problems in 4 pages, each has 25 points

1. Define \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) by \( f(0, 0) = 0 \) and

\[
f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}, \quad \text{if } (x, y) \neq (0, 0).
\]

Whether \( f \) is differentiable at \((0, 0)\)?
2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^1$ function. Prove that there is a continuous one-to-one function $g$ from $[0, 1]$ into $\mathbb{R}^2$ such that the composite function $f \circ g$ is constant.

Hint: We may assume $f$ is not constant, otherwise this is trivial. So there is $(x_0, y_0) \in \mathbb{R}^2$ such that $Df(x_0, y_0) \neq 0$. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x, y) = (f(x, y), y).$$

Show that $F$ has a local inverse $G$. Let $a = f(x_0, y_0)$ and $\gamma$ be a one-to-one map of $[0, 1]$ to a small enough neighborhood of $y_0$ in $\mathbb{R}$. Show that $g(t) = G(a, \gamma(t))$ has the desired property.
3. Let $\vec{F}$ be the vector field $\langle x^2 + y - 4, 3xy, 2xz + z^2 \rangle$ on $\mathbb{R}^3$. Compute $\nabla \times \vec{F}$ (the curl of $\vec{F}$) and the integral of $\nabla \times \vec{F}$ over the surface $H = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 16, \ z \geq 0 \}$.

Hint: Let $D = \{ (x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 16 \}$. Note that $H$ and $D$ have the same boundary. Apply Stoke’s theorem twice to transform an integral over $H$ to an integral over $D$. 
4. (Bonus Problem) Let $M(n)$ be the space of $n \times n$ matrices over $\mathbb{R}$, identified in the usual way with $\mathbb{R}^{n^2}$. Let the function $F$ from $M(n)$ into $M(n)$ be defined by

$$F(X) = X + X^2.$$  

Prove that the range of $F$ contains a neighborhood of the origin.

*Hint: We have*

$DF(X)Y = \lim_{h \to 0} \frac{F(X + hY) - F(X)}{h}$.

*Show that $DF(0)$ is invertible.*