Test 2

Note: Except perhaps for problem 4(b), all the answers are “one-liners” and problem 4(b) is not much longer.

1) Show that if \( f : [a, b) \to (a, b) \) is a bijection, then \( f((a, b)) \) is not connected. Deduce that \([a, b)\) and \((a, b)\) are not homeomorphic.

2) Let \( A \) and \( B \) be two compact subsets of the metric space \((M, d)\). Show that there are \( a_0 \in A \) and \( b_0 \in B \) such that \( d(a_0, b_0) \leq d(a, b) \) for every \((a, b) \in A \times B\). (Hint: compactness of \( A \times B \) and continuity of the distance \( d\)).

3) Let \((M, d_M)\) and \((N, d_N)\) be metric spaces with \( N \) discrete (for instance, \( N = \mathbb{N} \) or \( N = \mathbb{Z} \)) and let \( f : M \to N \) be continuous. Given any \( y \in N\), show that the set \( f^{-1} \{y\}\) is open and closed in \( M \) (hint: in a discrete space, points are open and closed). Deduce that if \( M \) is connected and \( f \) is continuous, then \( f \) is constant.

4) Let \((M, d)\) be a metric space. As usual, the graph \( \Gamma \) of a function \( f : M \to \mathbb{R} \) is defined by

\[
\Gamma = \{(x, f(x)) : x \in M \} \subset M \times \mathbb{R}.
\]

(a) Show that if \( f \) is continuous and \( M \) is compact, then \( \Gamma \) is compact.
(b) Show that if \( \Gamma \) is compact, then \( M \) is compact (trivial) and \( f \) is continuous.

5) Assume that \( f : \mathbb{R} \to \mathbb{R} \) is continuous, that \( f'(x) \) exists for every \( x \neq 0 \) and that \( \lim_{x \to 0} \frac{f'(x)}{x} = L \in \mathbb{R} \) exists. Show that \( f \) is differentiable at 0 and that \( f'(0) = L \). (Hint: mean-value theorem.)