1. Suppose that $f'$ exists and is decreasing on $[0, \infty)$ and $f(0) = 0$. Prove that $f(x)/x$ is decreasing on $(0, \infty)$.

2. Let $n \in \mathbb{N}$, $f = (f_1, \ldots, f_n) : \mathbb{R} \to \mathbb{R}^n$ be $C^1$. Suppose that $f(0) = 0$ and $Df(0) \neq 0$. Prove that there exists a $\delta > 0$ such that the function
   \[ |f(t)| = \left( \sum_{j=1}^{n} (f_j(t))^2 \right)^{1/2} \]
   is increasing on $(0, \delta)$.

3. Let $u : \mathbb{R}^n \to R$ and $\phi : R \to \mathbb{R}$ be two $C^2$ functions. Let $v(x) = \phi(u(x))$. Suppose that $u$ is harmonic and $\phi$ is convex. Prove that $\Delta v(x) \geq 0$ for $x \in \mathbb{R}^n$ (i.e. $v$ is subharmonic).

4. Let $v : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$ function such that $v$ is nonconstant on
   \[ \Omega = \{(x, y) : x^2 + y^2 \leq 1\} \]
   and $v(x, y) = 0$ for all $(x, y) \in \partial \Omega$. Suppose that for some $\lambda \in \mathbb{R}$ we have
   \[ \Delta v(x, y) = \lambda v(x, y) \]
   for every $(x, y) \in \Omega$. Prove that $\lambda < 0$.

5. Let $f = (u, v, w) : \mathbb{R}^3 \to \mathbb{R}^3$ be a $C^2$ mapping such that the first order partial derivatives of $u$, $v$ and $w$ are bounded on $\mathbb{R}^3$ and
   \[ |f(x, y, z)| \leq \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \]
   for every $(x, y, z) \in \mathbb{R}^3 \setminus \{0\}$. Prove that
   \[ \lim_{r \to \infty} \left( \iint_{B(0,r)} \frac{\partial(u, v, w)}{\partial(x, y, z)} dV \right) = 0. \]

6. Let $x = (x_1, x_2, x_3)$, $B_r = \{ x \in \mathbb{R}^3 : |x| \leq r \}$ and $f : B_1 \to \mathbb{R}$ be $C^1$. Prove that
   \[ \lim_{\delta \to 0^+} \iint_{B_1 \setminus B_\delta} \left( \frac{x \cdot \nabla f(x)}{|x|^3} \right) dV = \iint_{|x|=1} f(x) dS - 4\pi f(0, 0, 0) \]
7. Let $f(x) = f(x_1, x_2)$ be a $C^2$ function on $\mathbb{R}^2$ such that $f(0) = 0$ and $\nabla f(0) = 0$. Define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{x \cdot \nabla f(x)}{|x|^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that $g$ is Riemann integrable on $\{x \in \mathbb{R}^2 : |x| \leq 1\}$ and

$$\int \int_{|x|\leq1} g \, dA = \int_{|x|=1} f \, ds.$$ 

8. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $f \in C^\infty(\mathbb{R}^2)$. Suppose that $f(x, y) = 0$ for all $(x, y) \in \partial D$. Prove that

$$|f(x_1, y_1) - f(x_2, y_2)| \leq 10|(x_1, y_1) - (x_2, y_2)|$$

holds for all $(x_1, y_1), (x_2, y_2) \in D$.

9. Prove that $f(x) = \sqrt{x}$ is Lipschitz on $[1, \infty)$ but not Lipschitz on $(0, 1]$.

10. Let $D = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$ and $f : D \to \mathbb{R}$ be differentiable. Suppose that $|\nabla f(x, y)| \leq 1$ holds for all $(x, y) \in D$. Prove that

$$|f(x_1, y_1) - f(x_2, y_2)| \leq 10|(x_1, y_1) - (x_2, y_2)|$$

holds for all $(x_1, y_1), (x_2, y_2) \in D$.

11. Let $\omega$ be the 1-form in $\mathbb{R}^3 \setminus \{(t, t, t) : t \in \mathbb{R}\}$ defined by

$$\omega = \frac{z-y}{(x-z)^2 + (y-z)^2} \, dx + \frac{x-z}{(x-z)^2 + (y-z)^2} \, dy + \frac{y-x}{(x-z)^2 + (y-z)^2} \, dz.$$ 

Show that $\omega$ is closed but not exact.

12. Suppose that $f : \mathbb{R}^2 \to [0, \infty)$ is uniformly continuous on $\mathbb{R}^2$ and

$$\sup_{r>0} \left( \int \int_{x^2+y^2 \leq r^2} f(x, y) \, dA \right) < \infty.$$ 

Prove that $\lim_{|(x, y)| \to \infty} f(x, y) = 0$.

13. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and let $u$ be a nonconstant real-valued $C^2$ function on a neighborhood of $D$ which satisfies $u(x, y) = 0$ for all $(x, y) \in \partial D$. Prove that

$$\int \int_D u \Delta u \, dA < 0.$$
14. Suppose that \( f \in C^2(\mathbb{R}^n \setminus \{0\}) \) depends on \( r = |x| \) only, i.e. \( f(x) = g(|x|) = g(r) \) for some \( g \in C^2(0, \infty) \). Express the Laplace operator

\[
\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x)
\]

in terms of \( n, r, g \) and derivatives of \( g \) only.

15. Prove that if \( K \in C^1(\mathbb{R} \setminus \{(0,0)\}) \) satisfies the estimate

\[
|\nabla K(x)| \leq \frac{1}{|x|^3}
\]

for all \( x \neq (0,0) \)

then there is a constant \( C > 0 \) such that

\[
\iint_{\{x \in \mathbb{R}^2 : |x| > 2|y|\}} |K(x - y) - K(x)| \, dx \leq C
\]

for all \( y \in \mathbb{R}^2 \).

**Hint:** Use the mean value theorem to estimate \( |K(x - y) - K(x)| \) and then integrate in polar coordinates.