Math 1540 Advanced Calculus 2
Homework #4 (Due Friday, Mar. 20, 2015)

1. If \( \{f_n\} \) is a sequence of measurable functions, prove that the set of points \( x \) at which \( \{f_n(x)\} \) converges is measurable.

2. If a bounded function \( f \) on \( [a, b] \) is Riemann integrable and if \( F(x) = \int_a^x f(t) dt \), prove that \( F'(x) = f(x) \) almost everywhere on \( [a, b] \).

3. Suppose \( f \in L^2(\mu), g \in L^2(\mu) \). Prove that
\[
\left|\int f \bar{g} \, d\mu\right|^2 = \int |f|^2 \, d\mu \int |g|^2 \, d\mu
\]
if and only if there is a constant \( c \) such that either \( g(x) = cf(x) \) or \( f(x) = cg(x) \) almost everywhere.

4. Suppose \( \{n_k\} \) is an increasing sequence of positive integers and \( E \) is the set of all \( x \in (-\pi, \pi) \) at which \( \{\sin n_k x\} \) converges. Prove that \( m(E) = 0 \).

   \textit{Hint: For every } A \subset E, \int_A \sin n_k x \, dx = 0, \text{ and } 
\[2 \int_A (\sin n_k x)^2 \, dx = \int_A (1 - \cos 2n_k x) \, dx \to m(A) \quad k \to \infty.\]

5. Prove that if \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a Lipschitz function, i.e. \(|f(x) - f(y)| \leq L|x - y|\) for all \( x, y \in \mathbb{R}^n \) and some \( L > 0 \) and \( E \subset \mathbb{R}^n \) is a set of measure zero, then \( f(E) \subset \mathbb{R}^n \) has measure zero.

6. Assume that \( f : [0, 1] \to [0, 1] \) is a continuous function such that the set \( \{x \in [0, 1] : f(x) = 1\} \) has measure zero. Prove directly (without using any results like monotone or dominated convergence theorem) that
\[
\lim_{n \to \infty} \int_0^1 f(x)^n \, dx = 0.
\]

7. Let \( f : [0, 1]^2 \to [0, \infty) \) be Riemann integrable over \([0,1]^2\). Suppose that
\[
\int_{[0,1]^2} f = 0.
\]
Prove that \( \{(x, y) \in [0,1]^2 : f(x,y) > 0\} \) is a set of measure zero.
8. Suppose \( \{f_n\} \) is a sequence of functions in \( L^1(\mu) \) and \( f_n \to f \) uniformly. If \( \mu(X) < \infty \), then \( f \in L^1(\mu) \) and \( \int f_n \to \int f \).

9. If \( f \in L^1(m) \) and \( F(x) = \int_{-\infty}^{x} f(t)dt \), then \( F \) is continuous on \( \mathbb{R} \). Here \( m \) is the Lebesgue measure.
   
   *Hint: Use Dominated Convergence Theorem.*

10. Let \( \mu \) be counting measure on the set \( \mathbb{N} \) of all natural numbers. Interpret Fatou’s Lemma and the monotone convergence theorem as statements about infinite series.

11. Let \( f(x) = x^{-1/2} \) if \( 0 < x < 1 \) and \( f(x) = 0 \) otherwise. Let \( \{r_n\} \) be an enumeration of the rational numbers, and set
   
   \[ g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n). \]
   
   Prove
   
   (a) \( g \in L^1(m) \) and compute \( \int_{-\infty}^{\infty} g(x)dx \).
   (b) \( g < \infty \) a.e.
   (c) \( g \) is discontinuous at each point and unbounded on any interval \((a, b)\).
   (d) \( g^2 \) is not integrable on any interval.
   
   *Hint: For (a) and (c), use Monotone Convergence Theorem.*

12. Suppose \( \mu(X) < 1 \). If \( f \) and \( g \) are complex-valued measurable functions on \( X \), define
   
   \[ \rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu. \]
   
   Then \( \rho \) is a metric on the space of measurable functions (we identify functions that are equal a.e.), and \( f_n \to f \) with respect to this metric if and only if \( f_n \to f \) in measure.