Math 1530 Advanced Calculus 1

Homework #5 (Due Monday, Nov. 17, 2014)

Note: $f \in \mathcal{R}$ denotes that $f$ is Riemann integrable.

$f \in \mathcal{R}(\alpha)$ denotes that $f$ is Riemann-Stieltjes integrable w.r.t. $\alpha$.

1. Suppose $f$ is a continuous real valued function. Show that
$$\int_0^1 f(x)x^2 dx = \frac{1}{3} f(\xi)$$
for some $\xi \in [0, 1]$.

2. Suppose $\alpha$ increases on $[a, b]$, $a \leq x_0 \leq b$, $\alpha$ is continuous at $x_0$, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$ Prove that $f \in \mathcal{R}(\alpha)$ and that $\int_a^b f \, d\alpha = 0$.

3. Suppose $f \geq 0$, $f$ is continuous on $[a, b]$, and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

4. If $f(x) = 0$ for all irrational $x$, $f(x) = 1$ for all rational $x$, prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

5. Suppose $f$ is a bounded real function on $[a, b]$ and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?