Problem 1. Let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be the matrix of a linear mapping $A \in L(\mathbb{R}^n; \mathbb{R}^m)$. Prove that the operatorial norm,

$$||A|| = \sup_{||x||=1} ||Ax||$$

satisfies the inequality

$$||A|| \leq ||A||_{HS} := \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

Moreover, if $A$ and $B$ are $n \times n$ matrices, prove that

$$||AB||_{HS} \leq ||A||_{HS}||B||_{HS}.$$ 

Problem 2. 

(a) Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \begin{cases} 
1 - \cos((x + y)^2) / (x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}$$

is continuous.

(b) Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \begin{cases} 
x^2 + y^2 - 2x^2y + \frac{4x^6y^2}{(x^4 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}$$

is continuous.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = f(xy)$. Prove that

$$x \frac{\partial F}{\partial x} = y \frac{\partial F}{\partial y}.$$ 

Problem 4. Suppose that $f \in C^2(\mathbb{R}^n - \{0\})$ depends on $r = |x|$ only, i.e. $f(x) = g(|x|) = g(r)$ for some $g \in C^2((0, \infty))$. Express the Laplace operator
\[ \Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x) \]

in terms of \( n, r, g \) and derivatives of \( g \) only.

University of Pittsburgh Preliminary Examination, 2009

**Problem 5.** Decide which of the following functions are differentiable at \((0,0)\):

(a) \[
 f(x,y) = \begin{cases} 
 \frac{x^3 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\
 0, & (x,y) = (0,0)
\end{cases}
\]

(b) \[
 f(x,y) = \begin{cases} 
 |xy|^{3/2} y / x^2 + y^4, & (x,y) \neq (0,0) \\
 0, & (x,y) = (0,0)
\end{cases}
\]

**Problem 6.**

(a) Write down an example of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that the directional derivative \( f_u(0,0) \) exists in \( \mathbb{R} \) for all unit vectors \( u \in \mathbb{R}^2 \), and yet \( f \) is not differentiable at \((0,0)\). Also, prove these two facts for your example.

(b) Consider the function \( g : \mathbb{R}^2 \to \mathbb{R} \) given by

\[ g(x,y) = x^{2/3} y^{2/3} \]

for all \((x,y) \in \mathbb{R}^2\). Prove that \( g \) is differentiable at \((0,0)\).

University of Pittsburgh Preliminary Examination, 2005

**Problem 7.** Show that the vector field \( F(x) = x|x|^{-n} \) defined on \( \mathbb{R}^n - \{0\} \) is divergence free, i.e.

\[ \text{div} \ F(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{x_i}{|x|^n} \right) = 0, \]

for all \( x \neq 0 \).

**Problem 8.** Let \( D \) be a non-empty, open and connected subset of \( \mathbb{R}^m \) and \( f : D \to \mathbb{R}^n \) is such that there exists \( \alpha > 1 \) and \( L > 0 \) with

\[ ||f(x) - f(y)|| \leq L \cdot ||x - y||^\alpha, \]

for all \( x, y \in D \). Show that \( f \) is constant.

**Problem 9.** Let \( M_{n \times n} \) denote the vector space of real \( n \times n \) matrices. Define the map \( f : M_{n \times n} \to M_{n \times n} \) by \( f(X) = X^2 \). Find the derivative of \( f \).

Berkeley Preliminary Exam

**Problem 10.** Find all points \((x, y) \in \mathbb{R}^2\) where the function
$f(x, y) = |e^x - e^y|(x + y - 2)$

is differentiable.