Problem 1. Let \( f : [0, 1] \to \mathbb{R} \) be a function. Show that \( f \) is continuous if and only if the graph of \( f \) is a compact set in \( \mathbb{R}^2 \).

University of California Davis Preliminary Exam, 2010

Problem 2. Show that there is a unique continuous real-valued function \( f : [0, 1] \to \mathbb{R} \) such that
\[
f(x) = \sin x + \int_0^1 \frac{f(y)}{e^x + y+1} \, dy.
\]

Berkeley Preliminary Exam, 1984

Problem 3. Let \((X, d)\) be a nonempty complete metric space. Let \( S : X \to X \) be a given mapping and write \( S^2 \) for \( S \circ S \), i.e. \( S^2(x) = S(S(x)) \). Suppose that \( S^2 \) is a contraction. Show that \( S \) has a unique fixed point.

Berkeley Preliminary Exam, 1998

Problem 4. Let \((X, d)\) be a compact metric space and \( f : X \to X \) be a contractive mapping, i.e for all \( x, y \in X \), \( x \neq y \),
\[
d(f(x), f(y)) < d(x, y).
\]
Prove that \( f \) has unique fixed point in \( X \).

University of Pittsburgh Preliminary Examination, 2010

Problem 5. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous and surjective.

(a) Is \( f \) open?
(b) Show that \( f \) is open if and only if \( f \) is an homomorphism.

Problem 6. (a) Show that any monotonic and onto function \( f : \mathbb{R} \to \mathbb{R} \) is continuous.
(b) Show that any continuous and one-to-one function \( f : \mathbb{R} \to \mathbb{R} \) is monotonic.
(c) Find all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(f(x)) = -x \), for all \( x \in \mathbb{R} \).
**Problem 7.** Let \( f : (-\infty, \infty) \to \mathbb{R} \) be continuous and \( \lim_{x \to \infty} f(f(x)) = \infty \). Prove that \( \lim_{x \to \infty} |f(x)| = \infty \).

University of Pittsburgh Preliminary Exam, 2015

**Problem 8.** (a) Prove that any continuous mapping from the unit interval into the unit interval has a fixed point.
(b) [Knaster] Prove that any nondecreasing function \( f : [a, b] \to [a, b] \) has a fixed point.

**Problem 9.** Prove that any \( \alpha \)-Holder continuous function \( f : \mathbb{R} \to \mathbb{R} \), with \( \alpha > 1 \) is constant.

University of Lincoln-Nebraska Qualifying Exam, 1999

**Problem 10.** Suppose that \( f : [0, 1] \to \mathbb{R} \) is continuous and has local maximum at each point in \([0, 1]\). Prove that \( f \) is constant.

Ohio State University Qualifying Exam, 2005