Problem 1. Suppose that a $C^2$ function $f : \mathbb{R}^n \to \mathbb{R}$ has a local maximum at $x = 0$. Prove that for any $x \in \mathbb{R}^n$,

$$f(x) = f(0) + \int_0^1 (1-t)(D^2 f(tx))(x) dt.$$ 

Problem 2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ be two differentiable functions on $\mathbb{R}^n$. Suppose that $f(0) = 0$, $D(g \circ f)(0) = 0$, and $\det(Jf(0)) \neq 0$. Prove that $Dg(0) = 0$.

Problem 3. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Let $\Delta$ be the set of points $x \in \mathbb{R}^n$ satisfying conditions

$$\sum_{i=1}^n a_ix_i = 1, x_i > 0, i = 1, 2, \ldots, n.$$ 

Prove that the function $\log \left( \prod_{i=1}^n x_i \right)$ has a unique maximum on $\Delta$ and find the point where it occurs.

Problem 4. Given $a_1, a_2, \ldots, a_n$ real numbers, find the maximum value of $\left| \sum_{i=1}^n a_ix_i \right|$ on the unit sphere $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$.

Problem 5. Let $f \in C^2(\mathbb{R}^2)$. Suppose that $\nabla f = 0$ on a compact set $E \subset \mathbb{R}^2$. Prove that there is a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|^2,$$ 

for all $x, y \in E$. 

Problem 6. Prove that there exists a $\delta > 0$ such that, for each $(a, b) \in \mathbb{R}^2$ satisfying $|(a, b)| < \delta$, the equation

$$\int_0^1 (2t \cdot e^{t^3(a^2+b^2+x)} - e^{t^2(a^2+b^2-x^2)})dt = 0$$

has a solution $x \in (-1, 1)$.

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Problem 7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be $C^1$ functions on $\mathbb{R}$ with $f(0) = 0$ and $f'(0) \neq 0$. Prove that there exists a $\delta > 0$ and a $C^1$ function $\varphi : (-\delta, \delta) \to \mathbb{R}$ such that $\varphi(0) = 0$ and

$$(\cos x)f(\varphi(x)) = (\sin x)g(\varphi(x)),$$

for all $x \in (\delta, \delta)$.

Problem 8. Let $M_2(\mathbb{R})$ be the space of $2 \times 2$ matrices over $\mathbb{R}$, identified in the usual way with $\mathbb{R}^4$. Let the function $f : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $f(X) = X + X^2$ Prove that the range of $f$ contains a neighborhood of the origin.

Problem 9. Prove that there exists positive numbers $p, q > 0$ such that there are unique functions $u : (-1-p, -1+p) \to (1-q, 1+q)$, $v : (-1-p, -1+p) \to (1-q, 1+q)$ such that $u(-1) = v(-1) = 1$, and

$$xe^{u(x)} + u(x)e^{v(x)} = 0 = xe^{v(x)} + v(x)e^{u(x)},$$

for all $x \in (-1-p, -1+p)$.

Problem 10. Consider a function $f(x, y)$ which is twice continuously differentiable. Suppose that $f$ has its unique minimum at $(x, y) = (0, 0)$. Carefully prove that at $(0, 0)$,

$$\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} \geq \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2.$$

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