Problem 1. Evaluate
\[ \int \int_{D} e^{-x^2-y^2} dxdy, \]
where \( D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \).

Problem 2. Let \( D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \) and let \( u \) be a non-constant real-valued \( C^2 \) function on a neighborhood of \( D \) which satisfies \( u(x, y) = 0 \) for all \((x, y) \in \partial D\). Prove that
\[ \int \int_{D} u \Delta u dA < 0. \]

University of Pittsburgh Preliminary Examination, 2009

Problem 3. Prove that
\[ \int_{0}^{1} \int_{0}^{1} \frac{1}{1-xy} dxdy = \sum_{n=1}^{\infty} \frac{1}{n^2}. \]

Problem 4. Let \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( |x| = \sqrt{x_1^2 + x_2^2} \). Consider \( D = \{x \in \mathbb{R}^2 : |x| \leq 1\} \) and \( f : D \rightarrow \mathbb{R} \) be continuous on \( D \). Show that
\[ \lim_{n \to \infty} \int \int_{D} (n + 2)|x|^n f(x) dA = \int_{0}^{2\pi} f(\cos t, \sin t) dt. \]

University of Pittsburgh Preliminary Examination, 2011

Problem 5. Let \( u : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^2 \) function such that \( u \) is non-constant on the unit disc \( D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \) and \( u = 0 \) on \( \partial D \). Suppose that for some \( \lambda \in \mathbb{R} \) we have
\[ -\Delta u(x, y) = \lambda u(x, y), \]
for all \((x, y) \in D\). Show that \( \lambda > 0 \).
Berkeley Preliminary Examination, 1994

Problem 6. Let $\lambda, a \in \mathbb{R}$, $a < 0$ and $u(x, y)$ be an infinitely differentiable function defined on an open neighborhood of closed unit disc $D$ such that

$$\begin{align*}
-\Delta u &= - \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \lambda u \quad \text{in} \quad D \\
D_n u &= au \quad \text{on} \quad \partial D
\end{align*}$$

Here $D_n u$ denotes the directional derivative of $u$ in the direction of the outward unit normal. Prove that if $u$ is not identically zero in the interior of the domain $D$, then $\lambda > 0$.

Berkeley Preliminary Examination, 2003

Problem 7. Let $D$ be the closed unit disc centered at $(0, 0)$ in $\mathbb{R}^2$. Prove that if $f : D \to \mathbb{R}$ is a continuous function, then

$$\left| \int \int_D e^{x^2 + y^2} f(x, y)dxdy \right| \leq \sqrt{\frac{\pi}{2}} (e^2 - 1) \cdot \left( \int \int_D f^2(x, y)dxdy \right)^{1/2}.$$  

University of Missouri-Columbia Qualifying Examination, 2002

Problem 8. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $f \in C^\infty(\mathbb{R}^2)$. Suppose that $f(x, y) = 0$ for all $(x, y) \in \partial D$. Prove that

$$\left| \int \int_D f(x, y)dA \right| \leq \sqrt{\frac{\pi}{8}} \left( \int \int_D |\nabla f(x, y)|^2 dA \right)^{1/2}.$$

Problem 9. Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ and $f \in C^\infty(\mathbb{R}^3)$. Suppose that $f(x, y, z) = 0$ for all $(x, y, z) \in \partial D$. Prove that

$$\left| \int \int \int_{\Omega} f(x, y, z)dV \right| \leq \frac{2\sqrt{5\pi}}{15} \left( \int \int \int_{\Omega} |\nabla f(x, y, z)|^2 dV \right)^{1/2}.$$

University of Pittsburgh Preliminary Examination, 2013

Problem 10. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and $u \in C^2(\Omega)$ be such that $u = 0$ on $\partial \Omega$. Prove that for each $\epsilon > 0$ there holds true

$$\int_{\Omega} |\nabla u(x)|^2 dx \leq \epsilon \int_{\Omega} (\Delta u(x))^2 dx + \frac{1}{4\epsilon} \int_{\Omega} u^2(x) dx.$$