Problem 1. Evaluate

\[ \int \int_D e^{-x^2-y^2} \, dx \, dy, \]

where \( D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \).

Problem 2. Let \( \Omega \subset \mathbb{R}^3 \) be open, and suppose \( u, v \in C^2(\Omega) \). Consider the following vector field

\[ F = u \nabla v. \]

(i) Prove that

\[ \text{div} \, F = \nabla \cdot F = u \Delta v + \nabla u \nabla v, \]

where \( \Delta v = \nabla \cdot (\nabla v) = \text{div}(\nabla v) = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} \) is the Laplace operator.

(ii) If \( \Omega \) is bounded with smooth boundary, prove the Green’s first identity,

\[ \int_{\Omega} u \Delta v \, dV + \int_{\Omega} \nabla v \nabla u \, dV = \int_{\partial \Omega} u \frac{\partial v}{\partial \nu} \, dA, \]

where \( \frac{\partial v}{\partial \nu} = \nabla v \cdot \nu \) is the outward normal derivative (Thus \( \frac{\partial v}{\partial \nu} \) is the directional derivative of \( v \) in the direction of the outward normal to \( \partial \Omega \)).

(iii) Interchange \( u \) and \( v \), subtract the resulting formula from the first one, to obtain Green’s second identity,

\[ \int_{\Omega} (u \Delta v - v \Delta u) \, dV = \int_{\partial \Omega} \left( u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \, dA. \]

(iv) Assume \( v \) is harmonic and \( u = 1 \) on \( \Omega \). Show that

\[ \int_{\partial \Omega} \frac{\partial v}{\partial \nu} \, dA = 0. \]
Problem 3. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and let $u$ be a non-constant real-valued $C^2$ function on a neighborhood of $D$ which satisfies $u(x, y) = 0$ for all $(x, y) \in \partial D$. Prove that

$$\int \int_D u \Delta u dA < 0.$$  

University of Pittsburgh Preliminary Examination, 2009

Problem 4. Suppose $f \in C^2(\mathbb{R}^3)$ is constant in a neighborhood of the boundary of the ball $B \subseteq \mathbb{R}^3$. Prove that

$$\int \int \int_B (f_{xx} + f_{yy} + f_{zz}) dV = 0.$$  

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Problem 5. Prove that

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} dxdy = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where the integral is understood as an improper integral $\lim_{t \to 1-} \int_0^t \int_0^t \ldots$

EXTRA CREDIT CHALLENGE !!! Deduce from here the Euler celebrated series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$  

Problem 6. Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $|x| = \sqrt{x_1^2 + x_2^2}$. Consider $D = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ and $f : D \to \mathbb{R}$ be continuous on $D$. Show that

$$\lim_{n \to \infty} \int \int_D (n + 2)|x|^n f(x) dA = \int_0^{2\pi} f(\cos t, \sin t) dt.$$  

University of Pittsburgh Preliminary Examination, 2011

Problem 7. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$ function such that $u$ is non-constant on the unit disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $u = 0$ on $\partial D$. Suppose that for some $\lambda \in \mathbb{R}$ we have

$$-\Delta u(x, y) = \lambda u(x, y),$$

for all $(x, y) \in D$. Show that $\lambda > 0$.

Berkeley Preliminary Examination, 1994
Problem 8. Let $Q \in C^\infty(\mathbb{R}^n)$ real-valued linear mapping such that $Qf \geq 0$ whenever $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(0) = 0$ and $f(x) \geq 0$ in a neighborhood of 0. Prove that there are real numbers $a_{ij}, b_i$ and $c$ such that

$$Qf = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) + \sum_{i=1}^{n} b_i \frac{\partial f}{\partial x_i}(0) + cf(0).$$

Problem 9. Let $D$ be the closed unit disc centered at $(0,0)$ in $\mathbb{R}^2$. Prove that if $f : D \to \mathbb{R}$ is a continuous function, then

$$\left| \int \int_{D} e^{x^2+y^2} f(x,y) dxdy \right| \leq \sqrt{\frac{\pi}{2}} (e^2 - 1) \cdot \left( \int \int_{D} f^2(x,y) dxdy \right)^{1/2}.$$

Problem 10. Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \}$ and $f \in C^\infty(\mathbb{R}^2)$. Suppose that $f(x,y) = 0$ for all $(x,y) \in \partial D$. Prove that

$$\left| \int \int_{D} f(x,y) dA \right| \leq \frac{\sqrt{\pi}}{8} \left( \int \int_{D} |\nabla f(x,y)|^2 dA \right)^{1/2}.$$