Problem 1. Decide if the following sequence of functions converge uniformly or not:

\[ f_n : [0, 1] \rightarrow \mathbb{R}, f_n(x) = \frac{x}{1 + n^2 x^2}, n \geq 1. \]

Problem 2. Prove that the series

\[ \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^2 + n}{n^2} \]

converges uniformly in every bounded interval, but does not converge absolutely for any value \( x \).

Problem 3. Let \( (f_n)_n \) be a uniformly bounded sequence of functions which are Riemann integrable on \([a, b]\), and put

\[ F_n(x) = \int_a^x f_n(t)dt, a \leq x \leq b. \]

Prove that there exists a subsequence \((F_{n_k})\) which converges uniformly on \([a, b]\).

Problem 4. Prove that, if \( f : [0, 1] \rightarrow \mathbb{R} \) is continuous and

\[ \int_0^1 x^n f(x)dx = 0 \]

for each integer \( n \geq 0 \), then \( f \equiv 0 \) on \([0, 1]\).

University of Pittsburgh Preliminary Exam, 2005

Problem 5. Suppose that \( f : [0, \infty) \rightarrow \mathbb{R} \) is continuous on \([0, \infty)\), differentiable on \((0, \infty)\), \( f(0) = 0, 0 \leq f'(x) \leq 1 \), for \( x > 0 \). Show that for \( x \geq 0, \)

\[ \left( \int_0^x f(t)dt \right)^2 \geq \int_0^x f^3(t)dt. \]

University of Missouri-Columbia Qualifying Exam, 2007
Problem 6. Let $a \in [0, 1]$. Show that there is no continuous function $f : [0, 1] \to (0, \infty)$ such that
\[
\int_0^1 f = 1, \\
\int_0^1 xf = a, \\
\int_0^1 x^2 f = a^2.
\]

Problem 7. Suppose $f$ is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and $\int_a^b f^2(x)dx = 1$. Prove that
\[
\int_a^b x f(x)f'(x)dx = -\frac{1}{2},
\]
and that
\[
\int_a^b (f'(x))^2 dx \int_a^b x^2 f^2(x)dx \geq \frac{1}{4}.
\]

Problem 8. (a) [Rogers-Holder] Let $C([0, 1])$ denote the set of continuous real-valued functions on $[0, 1]$. Let $p, q > 0$, with $\frac{1}{p} + \frac{1}{q} = 1$. If $f$ and $g$ are in $C([0, 1])$, then
\[
\int_0^1 |f(x)g(x)|dx \leq \left(\int_0^1 |f(x)|^p dx\right)^{1/p} \left(\int_0^1 |g(x)|^q dx\right)^{1/q}.
\]

(b) [Minkovski] Let $p \in (1, \infty)$. For each function $f \in C([0, 1])$, let $\|f\|_p := \left(\int_0^1 |f(x)|^p dx\right)^{1/p}$. Prove that
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

Problem 9. (a) Show that
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n} = \int_0^1 x^e dx.
\]

Putnam Competition, 1969

(b) Show that
\[
\sum_{n=1}^{\infty} \frac{1}{n^n} = \int_{0}^{1} \frac{1}{x^x} \, dx.
\]

University of Wisconsin-Madison Qualifying Exam, 2013

**Problem 10.** [Polya] Let \( f \in C([a,b]) \cap C^1([a,b]) \) such that \( f(a) = f(b) = 0 \). Show that

\[
\int_{a}^{b} |f(x)| \, dx \leq \frac{(b-a)^2}{4} \sup_{x \in [a,b]} |f'(x)|.
\]