Problem 1. Let $f(x) = \sum_{k=1}^{n} a_k \sin(kx)$, with $a_1, a_2, \ldots, a_n \in \mathbb{R}$, $n \geq 1$. Prove that if $f(x) \leq |\sin x|$, for all $x \in \mathbb{R}$, then
\[
\left| \sum_{k=1}^{n} k a_k \right| \leq 1.
\]

Putnam A1, 1967

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Prove that $f$ is constant.

Problem 3. Let $f : [a, b] \to \mathbb{R}$ a function, continuous on $[a, b]$, and twice differentiable on $(a, b)$. If $f(a) = f(b)$ and $f'(a) = f'(b)$, prove that for every real number $\lambda$, the equation
\[
f''(x) - \lambda(f'(x))^2 = 0
\]
has at least one solution in the interval $(a, b)$.

Problem 4. Does there exist a continuously differentiable function $f : \mathbb{R} \to (0, \infty)$ such that $f'(x) = f(f(x))$ for all $x$?

International Mathematics Competition, 2002

Problem 5. Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all $x$?

Putnam B5, 2010

Problem 6. Let $c \geq 0$. Give a complete description, with proof, of a set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(x^2 + c)$ for all real $x$.

Putnam A6, 1986