ON A ERDOS INSCRIBED TRIANGLE INEQUALITY REVISITED

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Abstract. In this note we give a refinement of an inequality obtained by Torrejon [10] between the area of a triangle and that of an inscribed triangle. Our approach is based on using complex numbers and some elementary facts on geometric inequalities.

1. Introduction and Main result

Let us consider a triangle $ABC$ on each of the sides $BC$, $CA$ and $AB$ and fix arbitrary points $A_1, B_1, C_1$. As pointed out in [10], [7] a question with a long history is the following Erdos-Debrunner inequality:

\[ \min\{\text{area}(AC_1B_1); \text{area}(C_1BA_1); \text{area}(B_1AC)\} \leq \text{area}(A_1B_1C_1). \]  


\[ \mathcal{M}_{-1}\{\text{area}(AC_1B_1); \text{area}(C_1BA_1); \text{area}(B_1AC)\} \leq \text{area}(A_1B_1C_1), \]

where $\mathcal{M}_{-1}$ denotes the harmonic mean of the areas of triangles mentioned in the above inequality. Moreover, Janous formulated a more general question which is extended and solve by Mascioni [7], [8]. Using a different method, Frenzen, Ionaşcu and Stănică [5] proved Janous conjectures independently of Mascioni.

The purpose of this note is to extend the result obtained by Torrejon [10] regarding the areas of triangles $A_1B_1C_1$ and $ABC$ when the points $A_1, B_1, C_1$ satisfy a certain metric property. In fact our main result is given by the following

**Theorem 1.1.** Let $ABC$ be a triangle and let $A_1, B_1, C_1$ be on $BC = a$, $CA = b$ and $AB = c$ respectively with none of $A_1, B_1, C_1$ coinciding with a vertex of $ABC$. If

\[
\frac{AB + BA_1}{AC + CA_1} = \frac{BC + CB_1}{AB + AB_1} = \frac{AC + AC_1}{BC + BC_1} = \alpha,
\]

then

\[
\text{area}(A_1B_1C_1) \leq \frac{9abc}{4(a + b + c)(a^2 + b^2 + c^2)} \left( \text{area}(ABC) + s^4 \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 \text{area}(ABC)^{-1} \right),
\]

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where $s$ is the semi-perimeter of triangle $ABC$.

When $\alpha = 1$ we obtain

**Corollary 1.2. ([6])** Let $ABC$ be a triangle and let $A_1, B_1, C_1$ be on $BC = a$, $CA = b$ and $AB = c$ respectively with none of $A_1, B_1, C_1$ coinciding with a vertex of $ABC$. If

$$
\frac{AB + BA_1}{AC + CA_1} = \frac{BC + CB_1}{AB + AB_1} = \frac{AC + AC_1}{BC + BC_1} = \alpha,
$$

then

$$
\frac{\text{area}(A_1B_1C_1)}{\text{area}(ABC)} \leq \frac{9abc}{4(a + b + c)(a^2 + b^2 + c^2)}.
$$

Clearly, by the Arithmetic-Geometric mean inequality, we have

$$(a + b + c)(a^2 + b^2 + c^2) \geq 9abc,$$

and by Theorem 1.1 we obtain

**Theorem 1.3. ([10])** Let $ABC$ be a triangle and let $A_1, B_1, C_1$ be on $BC, CA, AB$, respectively, with none of $A_1, B_1, C_1$ coinciding with a vertex of $ABC$. If

$$
\frac{AB + BA_1}{AC + CA_1} = \frac{BC + CB_1}{AB + AB_1} = \frac{AC + AC_1}{BC + BC_1} = \alpha,
$$

then

$$
4 \text{area}(A_1B_1C_1) \leq \text{area}(ABC) + s^4 \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 \text{area}(ABC)^{-1},
$$

where $s$ is the semi-perimeter of triangle $ABC$.

When $\alpha = 1$ we derive Aassila’s inequality

**Corollary 1.4. ([1])** Let $ABC$ be a triangle, and let $A_1, B_1, C_1$ be on $BC, CA, AB$, respectively, with none of $A_1, B_1, C_1$ coinciding with a vertex of $ABC$. If

$$
AB + BA_1 = AC + CA_1,
$$

$$
BC + CB_1 = AB + AB_1,
$$

$$
AC + AC_1 = BC + BC_1,
$$

then

$$
4 \text{area}(A_1B_1C_1) \leq \text{area}(ABC).
$$

Our approach in computing the area of the triangle $A_1B_1C_1$ will be different from Torrejon’s one [10] and it is based on the geometry of complex numbers and combining with a straightforward geometric inequality we obtain the conclusion of the main result.
2. Proof of Theorem 1.1

First of all, we prove the following equality:

\[
\text{area}(A_1B_1C_1) = S \cdot \frac{2(s - a)(s - b)(s - c) + s^2 \cdot \left(\frac{a-1}{a+1}\right)^2(a + b + c)}{abc}.
\]

We use complex numbers. For simplicity denote by \(a, b, c\) the sidelengths of triangle \(ABC\), \(s\) its semiperimeter, \(S\) its area, \(z_A, z_B, z_C\) the affixes of the points \(A, B, C\) and by \(z_{A_1}, z_{B_1}, z_{C_1}\) the affixes of the points \(A_1, B_1, C_1\).

First of all, \(2s = a + b + c = (AB + AB_1) + (BC + CB_1) = (\alpha + 1)(c + AB_1)\) and, consequently \(AB_1 = \frac{2s}{\alpha + 1} - c\) and

\[CB_1 = CB - AB_1 = b - \frac{2s}{\alpha + 1} + c = 2s - a - \frac{2s}{\alpha + 1} = \frac{2s\alpha}{\alpha + 1} - a\]

Analogously we have

\[BC_1 = \frac{2s}{\alpha + 1} - a,\]
\[CA_1 = \frac{2s}{\alpha + 1} - b,\]
\[BA_1 = \frac{2s\alpha}{\alpha + 1} - c,\]
\[AC_1 = \frac{2s\alpha}{\alpha + 1} - b,\]

Denote \(z_{A_1}, z_{B_1}, z_{C_1}\) the affixes of \(A_1, B_1, C_1\), and they are given by

\[z_{A_1} = \frac{(2s_{\alpha+1} - b)z_B + (2s_{\alpha+1} - c)z_C}{a},\]
\[z_{B_1} = \frac{(2s_{\alpha+1} - c)z_C + (2s_{\alpha+1} - a)z_A}{b},\]
\[z_{C_1} = \frac{(2s_{\alpha+1} - a)z_A + (2s_{\alpha+1} - b)z_B}{c}.
\]

Now the formula for the area of triangle \(A_1B_1C_1\) is

\[2\text{area}(A_1B_1C_1) = \text{Im}\left(\sum_{cyc} z_{A_1}z_{B_1}\right) = \]
\[= \text{Im}\left(\sum_{cyc} \frac{(2s_{\alpha+1} - b)\overline{z_B} + (2s_{\alpha+1} - c)\overline{z_C}}{a}, \frac{(2s_{\alpha+1} - c)z_C + (2s_{\alpha+1} - a)z_A}{b}\right)\]
\[= \frac{1}{abc} \cdot \text{Im}\left(\sum_{cyc} \overline{z_B}z_C \left[c\left(\frac{2s}{\alpha + 1} - b\right)\left(\frac{2s}{\alpha + 1} - c\right) + b\left(\frac{2s\alpha}{\alpha + 1} - c\right)\left(\frac{2s\alpha}{\alpha + 1}\right)\right]\right)\]
\[ + \frac{1}{abc} \cdot \text{Im} \left( \sum_{\text{cyc}} z_C z_B a \left( \frac{2s}{\alpha + 1} - b \left( \frac{2s}{\alpha + 1} - c \right) \right) \right) \]

\[ = \frac{1}{abc} \cdot \text{Im} \left( \sum_{\text{cyc}} z_B z_C \left[ c \left( \frac{2s}{\alpha + 1} - b \left( \frac{2s}{\alpha + 1} - c \right) \right) + b \left( \frac{2s}{\alpha + 1} - c \right) - a \left( \frac{2s}{\alpha + 1} - b \right) \left( \frac{2s}{\alpha + 1} - c \right) \right] \right) \]

\[ = \frac{1}{abc} \cdot \text{Im} \left( \sum_{\text{cyc}} z_B z_C \left[ b \left( \frac{s(1 - \alpha)}{\alpha} + s - b \right) \left( \frac{s(1 - \alpha)}{\alpha} + s - c \right) + c \left( \frac{s(\alpha - 1)}{\alpha} + s - b \right) \left( \frac{s(\alpha - 1)}{\alpha} + s - c \right) \right] \right) \]

\[ - a \left( \frac{s(\alpha - 1)}{\alpha} + s - b \right) \left( \frac{s(1 - \alpha)}{\alpha} + s - c \right) \]

\[ = \frac{1}{abc} \cdot \text{Im} \left( \sum_{\text{cyc}} z_B z_C \left[ (s - b)(s - c)(b + c - a) + s \frac{1 - \alpha}{1 + \alpha} \left( b(s - b + s - c) - c(s - b + s - c) + a(s - b - s - c) \right) \right] \right) \]

\[ + s^2 \cdot \left( \frac{1 - \alpha}{1 + \alpha} \right)^2 (b + c + a) \]

\[ = \frac{1}{abc} \cdot \text{Im} \left( \sum_{\text{cyc}} z_B z_C \left[ 2(s - a)(s - b)(s - c) + s \frac{1 - \alpha}{1 + \alpha} (ab - ac + ac - ab) + s^2 \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 (a + b + c) \right] \right) \]

\[ = \frac{1}{abc} \cdot \text{Im} \left( \sum_{\text{cyc}} z_B z_C \left[ 2(s - a)(s - b)(s - c) + s^2 \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 (a + b + c) \right] \right) \]

\[ = \frac{2(s - a)(s - b)(s - c) + s^2 \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 (a + b + c)}{abc} \cdot \text{Im} \left( \sum_{\text{cyc}} z_B z_C \right) \]

which is equivalent to

\[ \text{area}(A_1 B_1 C_1) = S \cdot \frac{2(s - a)(s - b)(s - c) + s^2 \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 (a + b + c)}{abc}, \]

where we used the fact that \( z_A z_A \) and \( z_B z_C + z_C z_B \) are real numbers hence have the imaginary part 0.

Now we have that

\[ \frac{abc \cdot s}{2} \cdot \text{area}(A_1 B_1 C_1) = S \cdot s(s - a)(s - b)(s - c) + s^4 \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 \cdot S \]

Now using Heron’s formula we finally obtain

\[ \frac{abc \cdot s}{2} \cdot \text{area}(A_1 B_1 C_1) = S^3 + s^4 \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 \cdot S \]

In this moment, we are left to prove the following inequality

\[ \frac{sabc}{2} \geq 4S^2 \cdot \frac{(a + b + c)(a^2 + b^2 + c^2)}{9abc} \]
Using \( abc = 4RS \), we get the following equivalent inequality

\[
2sa^2b^2c^2 \geq 16S^2 \frac{(a + b + c)(a^2 + b^2 + c^2)}{9},
\]

which is successively equivalent to

\[
a^2b^2c^2 \geq 16S^2 \cdot \frac{a^2 + b^2 + c^2}{9},
\]

\[
9R^2 \geq a^2 + b^2 + c^2,
\]

which is evident since the distance between the circumcenter \( O \) and barycenter \( G \) is given by Leibniz identity \( OG^2 = 9R^2 - (a^2 + b^2 + c^2) \) and the proof of inequality (3) ends.

Now, we can conclude

\[
S^3 + s^4 \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 S \geq 4S^2 \cdot \frac{(a + b + c)(a^2 + b^2 + c^2)}{9abc} \text{ area}(A_1B_1C_1),
\]

which is finally equivalent to

\[
\text{area}(A_1B_1C_1) \leq \frac{9abc}{4(a + b + c)(a^2 + b^2 + c^2)} \left( S + s^4 \cdot \left( \frac{\alpha - 1}{\alpha + 1} \right)^2 \cdot S^{-1} \right)
\]

and thus our theorem is proved. □

**Remark.** In fact, inequality (3) can be rewritten in the following form:

\[
(b + c - a)(c + a - b)(a + b - c) \leq \frac{9a^2b^2c^2}{(a + b + c)(a^2 + b^2 + c^2)}.
\]

By Schur’s inequality,

\[
2(xy + yz + zx) - (x^2 + y^2 + z^2) \leq \frac{9xyz}{x + y + z},
\]

by putting \( x = a^2, y = b^2 \) and \( z = c^2 \), we have

\[
2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4) \leq \frac{9a^2b^2c^2}{a^2 + b^2 + c^2}
\]

which is equivalent to

\[
(a + b + c)(b + c - a)(c + a - b)(a + b - c) \leq \frac{9a^2b^2c^2}{a^2 + b^2 + c^2},
\]

which gives inequality (4).
REFERENCES


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