Problem 1. Find all numbers having odd number of positive divisors.

Problem 2. Is $4^9 + 6^{10} + 3^{20}$ a prime number?

Problem 3. (a) If $n$ is an integer number, show that $n^2 \equiv 0, 1$ or 4 (mod 8).
(b) Let $n$ and 24 be coprime. Show that $n^2 \equiv 1$ (mod 24).
(c*) Find the largest integer $a$ with the following property: if $n$ and $a$ are coprime, then $n^2 \equiv 1$ (mod $a$).

Problem 4. Let $a$ be a number of the form $4^k(8n+7)$. Show that $a$ cannot be written as the sum of three perfect squares.

Problem 5. Let $p$ be prime, $a$ be integer. Show that the number of solutions of the congruence $x^2 \equiv a$ (mod $p$) is equal to

\[
\left(\frac{a}{p}\right) + 1.
\]

Problem 6. Find all quadratic residues (a) Modulo 7; (b) Modulo 11.

Problem 7. (a) How many prime numbers of the form $4n + 2$ are there?
(b) Show that there are infinitely many prime numbers.
(c) Show that there are infinitely many prime numbers of the form $4k + 3$.
(d) Show that there are infinitely many prime numbers of the form $4k + 1$.

Problem 8. In this problem, $p > 2$ is a prime number, $a$ is an integer.
(a) How many solutions does the congruence $x^2 - y^2 \equiv a$ (mod $p$) have?
(b) Assume that $p \equiv 1$ (mod 4). How many solutions does the congruence $x^2 + y^2 \equiv a$ (mod $p$) have?
(c*) Assume that $p \equiv 3$ (mod 4). How many solutions does the congruence $x^2 + y^2 \equiv a$ (mod $p$) have?

Problem 9. Using quadratic reciprocity, calculate

\[
\left(\frac{113}{997}\right).
\]

Problem 10. Let $p > 2$ be a prime number, $n$ a positive integer. Show that the congruence

\[x^2 \equiv a \pmod{p^n}\]

has a solution if and only if

\[x^2 \equiv a \pmod{p}\]

has a solution.
Hint to Problem 1. Divisors come in pairs.

Hint to Problem 7b. Assume that \( p_1, \ldots, p_n \) are all prime numbers and consider the number \( p_1 \ldots p_n + 1 \).

Hint to Problem 7c. Assume that \( p_1, \ldots, p_n \) are all such numbers and consider the number \( 4p_1 \ldots p_n + 3 \).

Hint to Problem 7d. Use the theorem about divisors of \( a^2 + 1 \).

Hint to Problem 8a. Set \( u = x - y, v = x + y \).

Hint to Problem 8b. Use \(-1\) is a quadratic residue modulo \( p \).

Hint to Problem 8c. Relate the number of solutions to the number of solutions of \( x^2 + y^2 = az^2 \). Use the fact that \( a \) or \(-a\) is a quadratic residue modulo \( p \).

Hint to Problem 10. Show that if \( x \) is a solution of \( x^2 \equiv a \pmod{p^k} \), then for some \( a \) the number \( x + ap^k \) is a solution of \( x^2 \equiv a \pmod{p^{k+1}} \).