PUTNAM SEMINAR (MATH-1010), FALL 2017
HOMEWORK NO.

LECTURER: CEZAR LUPU

Problem 1. Show that
\[ 2^n \geq n^2 \]
for all \( n \geq 4 \), and
\[ \sum_{k=1}^{n} \frac{1}{k^2} < 2 \]
for all \( n \geq 1 \).

Problem 2. Let \( f \) be a polynomial of degree 2 with integer coefficients. Suppose that \( f(k) \) is divisible by 5 for every integer \( k \). Prove that all coefficients of \( f \) are divisible by 5.

IMC (International Mathematics Competition), 2007

Problem 3. Show that there does not exist a strictly increasing function \( f : \mathbb{N} \to \mathbb{N} \) satisfying \( f(2) = 3 \) and \( f(mn) = f(m)f(n) \) for all \( m, n \in \mathbb{N} \).

Problem 4. Prove that \( f(n) = 1 - n \) is the only integer-valued function defined on the integers that satisfies the conditions:

(i) \( f(f(n)) = n \) for all integers \( n \);
(ii) \( f(f(n + 2) + 2) = n \) for all integers \( n \);
(iii) \( f(0) = 1 \).

Putnam A1, 1992

Problem 5. Find all polynomials \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) such that
\[ P(x^2 + 1) = P(x)^2 + 1 \]
and \( P(0) = 0 \).

Putnam A2, 1971

Problem 6. Prove that for any distinct integers \( a_1, a_2, \ldots, a_n \) the polynomial
\[ P(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1 \]
cannot be written as a product of two nonconstant polynomials with integer coefficients.