Advanced Calculus & Real Analysis

I. Sequences & Series of Real Numbers

Definition of liminf and limsup.

Let \((a_n)_{n=1}^{\infty}\) be a sequence of real numbers. We define:

\[
\liminf_{n \to \infty} a_n = \lim_{n \to \infty} \left( \inf_{k \geq n} a_k \right)
\]

\[
\limsup_{n \to \infty} a_n = \lim_{n \to \infty} \left( \sup_{k \geq n} a_k \right).
\]

Stolz-Cesaro Theorem for liminf & limsup.

If \((b_n)_{n=1}^{\infty}\) is a sequence of positive real numbers such that \(\sum_{n=1}^{\infty} b_n = \infty\), then for any sequence \((a_n)_{n=1}^{\infty}\) in \(\mathbb{R}\) one has the inequalities:

1. \[
\limsup_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \limsup_{n \to \infty} \frac{a_n}{b_n}
\]

2. \[
\liminf_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \geq \liminf_{n \to \infty} \frac{a_n}{b_n}.
\]

In particular, if the sequence \((\frac{a_n}{b_n})_{n=1}^{\infty}\) has a limit, then

\[
\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = \lim_{n \to \infty} \frac{a_n}{b_n}.
\]

Proof:

It is enough to prove (1) since the other inequality follows from (1) by replacing \(a_n\) by \(-a_n\).

If \(\limsup_{n \to \infty} \frac{a_n}{b_n} = \infty\), then the left inequality is trivial. Suppose that \(\limsup_{n \to \infty} \frac{a_n}{b_n} = \infty\).
the quantity \( \limsup_{n \to \infty} \frac{a_n}{b_n} = L \) is either finite or \(-\infty\), and let us fix for the moment some number \( L \geq L \). By the definition of the \( \limsup \), there exists an index \( k \in \mathbb{N} \), such that

\[
\frac{a_n}{b_n} \leq L, \quad b_n > k.
\]

This implies that

\[
(*) \quad a_1 + a_2 + \ldots + a_n \leq a_1 + a_2 + \ldots + a_k + L(b_{k+1} + b_{k+2} + \ldots + b_n),
\]

\( \forall n > k. \)

For simplicity, denote

\( A_n = a_1 + a_2 + \ldots + a_n \) and \( B_n = b_1 + b_2 + \ldots + b_n \).

Then, the above inequality \( (*) \) reads as follows:

\[
A_n \leq A_k + L(B_n - B_k), \quad \forall n \geq k.
\]

This is equivalent to

\[
(**) \quad \frac{A_n}{B_n} \leq \frac{A_k}{B_k} + L - \frac{B_k}{B_n} = 1 + \frac{A_k - L B_k}{B_n}.
\]

Since \( B_n \to \infty \), by fixing \( k \) and taking the \( \limsup \)

\( \ln (**), \) we obtain

\[
\limsup_{n \to \infty} \left( \frac{A_n}{B_n} \right) \leq L.
\]

In other words, we have obtained the inequality

\[
\limsup_{n \to \infty} \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq L, \quad (\forall \; L \geq L)
\]

which in turn will force

\[
\limsup_{n \to \infty} \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq L = \limsup_{n \to \infty} \frac{a_n}{b_n}.
\]
Remark.

An equivalent formulation of the above theorem is what we call the Cesaro-Stolz's theorem:

If \((y_n)_{n \geq 1}\) is an increasing sequence with \(\lim_{n \to \infty} y_n = \infty\), then for any sequence \((x_n)_{n \geq 1}\), the following is true:

\[
\limsup_{n \to \infty} \frac{x_n}{y_n} \leq \limsup_{n \to \infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}},
\]

and

\[
\liminf_{n \to \infty} \frac{x_n}{y_n} \geq \liminf_{n \to \infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}.
\]

In particular, if the sequence \(\left(\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right)_{n \geq 1}\) has a limit, then

\[
\lim_{n \to \infty} \frac{x_n}{y_n} = \lim_{n \to \infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}.
\]

The proof is actually really easy!

Consider the sequences \((a_n)_{n \geq 1}\) and \((b_n)_{n \geq 1}\) defined by \(a_1 = x_1, b_1 = y_1\) and \(a_n = x_n - x_{n-1}, b_n = y_n - y_{n-1}\), \(n \geq 2\).

Since \(x_n = \sum_{k=1}^{n} a_k^k\) and \(y_n = \sum_{k=1}^{n} b_k^k\), everything becomes clear!
Let \( (x_n)_{n=1}^{\infty} \) be a nonnegative, decreasing sequence such that \( \sum_{n=1}^{\infty} x_n \) is convergent. Prove that \( \sum_{n=1}^{\infty} 2^n x_{2n} \) is also convergent.

**Solution.**

Denote \( a_n = a_1 + a_2 + \cdots + a_n \), \( n \geq 1 \), and \( b_n = a_1 + 2a_2 + \cdots + 2^n a_{2n} \), \( n \geq 1 \).

Note that \( a_{2n+1} = a_1 + a_2 + \cdots + a_{2n+1} \)

\[
= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + \underbrace{2\text{-terms}}_{4\text{-terms}}
+ (a_{2n} + \cdots + a_{2n+1})
\]

\[
\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots + \underbrace{2^2\text{-terms}}_{\text{am is decreasing}} + 2^n a_{2n} = b_n
\]

This means that if the sequence \( (b_n)_{n=1}^{\infty} \) is bounded, so is the sequence \( (a_n)_{n=1}^{\infty} \).

Now, let's prove the other way around!

\[
a_{2n} = a_1 + a_2 + \cdots + a_{2n} = a_1 + a_2 + (a_3 + a_3) +
+ (a_5 + a_6 + a_7 + a_8) + \cdots +
+ (a_{2n-1} + \cdots + a_{2n})
\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8)
+ \cdots + (a_{2n-2} + \cdots + a_{2n})
\geq \frac{1}{2} a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^n a_{2n}
\]
\[ \geq \frac{1}{2} (a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots + 2^m a_{2^n}) = \frac{1}{2} x_n \]
and therefore, if the sequence \((x_n)_{n \geq 1}\) is bounded, so is the sequence \((a_n)_{n \geq 1}\).

**Application 1.**

The harmonic p-series, \(\sum_{n=1}^{\infty} \frac{1}{n^p}\), \(p \in \mathbb{R}\), is convergent if \(p > 1\) and divergent for \(p \leq 1\).

Indeed, we have \(a_n = \frac{1}{n^p}\) and

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} 2^{n-p} \cdot \frac{1}{2^{np}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-p} = \sum_{n=1}^{\infty} \left(\frac{2^{1-p}}{2}\right)^n
\]

which is a geometric series. This series converges if and only if \(2^{1-p} < 1 = 2^0\), and thus means that \(1-p < 0\) or \(p > 1\). The series diverges when \(p \leq 1\).

**Application 2.**

The Bertrand series \(\sum_{n=2}^{\infty} \frac{1}{n \log n}\) is divergent.

Indeed, by the Cauchy condensation test, it follows that our series is equivalent with

\[
\sum_{n=2}^{\infty} \frac{2^n}{2^n \log 2^n} = \sum_{n=2}^{\infty} \frac{1}{n \log n} = \sum_{n=2}^{\infty} \frac{1}{n \log n} \approx \sum_{n=2}^{\infty} \frac{1}{n}
\]

which is divergent (see application 1).
Problem. (Olivier's criterion for series) (OSU Prelim 2003 & 2007)

Let \((a_n)_{n=1}^{\infty}\) be a sequence (decreasing) such that the series \(\sum_{n=1}^{\infty} a_n\) is convergent. Show that \(n a_n \to 0\).

Solution.

Since \(\sum_{n=1}^{\infty} a_n\) converges, it follows that \(a_n \to 0\) and since it is decreasing it follows that \(a_1 \geq a_2 \geq \ldots \geq a_n \geq 0\).

On the other hand, since \(\sum_{n=1}^{\infty} a_n\) converges, it follows that \(\sum_{k=n}^{\infty} a_k \to 0\).

Now, we have:

\[ 2\sum_{k=n}^{\infty} a_k \geq \sum_{k=n}^{2n} a_k \geq 2n a_{2n} \geq 0 \implies 2n a_{2n} \to 0 \]

\[ 2\sum_{k=n}^{\infty} a_k \geq \sum_{k=n}^{2n+1} a_k \geq (2n+1) a_{2n+1} \geq 0 \implies (2n+1) a_{2n+1} \to 0 \]
Second Solution.
We shall use Cauchy condensation test.

Since \( \sum_{n=1}^{\infty} a_n \) is convergent, by the Cauchy condensation test, it follows that the series \( \sum_{n=1}^{\infty} 2^n a_{2^n} \) is also convergent, and thus \( 2^n a_{2^n} \to 0 \) as \( n \to \infty \).

Now, consider the values of \( k \) for which \( 2^n < k < 2^{n+1} \). Because \( (a_n)_{n=1} \) is decreasing, we also have

\[ a_{2^{n+1}} \leq a_k \leq a_{2^n} \, . \]

Now, one can write

\[ 2^n \cdot a_{2^{n+1}} < k \cdot a_{2^{n+1}} \leq k \cdot a_k \leq k \cdot a_{2^n} < 2^{n+1} \cdot a_{2^n} \]

or equivalently

\[ \frac{1}{2} \left( 2^{n+1} a_{2^n+1} \right) \leq k \cdot a_k \leq 2 \cdot 2^n a_{2^n}. \]

By the squeeze theorem it follows that the term \( a_{2^n} \) in the middle goes to zero, so \( n a_n \to 0 \) as \( n \to \infty \). \( \square \)
Problem. (UCLA Basic Exam, 2009)
Set \( a_1 = 0 \) and define a sequence \( (a_n)_{n \geq 1} \) via the recurrence
\[
a_{n+1} = \sqrt{6 + a_n}, \quad \text{for all } n \geq 1.
\]
Show that this sequence converges and determine the limiting value.

Solution.
Firstly, we show by induction that \( 0 \leq a_n \leq a_{n+1} \leq 3 \) for all \( n \geq 1 \). Indeed, for \( n = 1 \), we have \( 0 \leq a_1 \leq a_2 \leq 3 \).
This is obvious since \( a_1 = 0 \) and \( a_2 = \sqrt{6} \). Now, suppose that \( 0 \leq a_n \leq a_{n+1} \leq 3 \) and we prove that
\[
0 \leq a_{n+1} \leq a_{n+2} \leq 3.
\]
Well, we have that \( a_{n+2} = \sqrt{6 + a_{n+1}} \geq \sqrt{6 + a_n} = a_{n+1} \).
Also, \( a_{n+2} = \sqrt{6 + a_{n+1}} \leq \sqrt{6 + 3} = 3 \) and \( a_{n+2} \geq \sqrt{6 + 0} = \sqrt{6} \geq 0 \).
Therefore, by induction it follows that
\[
0 \leq a_n \leq a_{n+1} \leq 3.
\]
By the monotone convergence theorem, it follows that
\[
\lim_{n \to \infty} a_n = L \quad \text{and by passing to the limit in our recursion, we get } \quad L = \sqrt{6 + L} \quad \text{or } \quad L^2 - L - 6 = 0 \quad \text{and this means that } \quad L = 3 \text{ or } L = -2.
\]
Since \( L \geq 0 \), it follows that \( L = 3 \) and we are done. \( \square \)
Problem. (OSU Qualifying Exam, 2010 & Pitt Prelim, 2011)

Let \((a_n)_{n=1}^{\infty}\) be a sequence of real numbers such that the series \(\sum_{n=1}^{\infty} a_n\) converges. Prove that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k \cdot a_k = 0.
\]

Solution.

Since the series \(\sum_{n=1}^{\infty} a_n\) converges, it follows that \(a_n \to 0\) as \(n \to \infty\). Also, by the Cauchy criterion, it implies that there exists a \(N_1 \in \mathbb{N}\) such that for all \(n \geq N_1\), we have:

\[
\left| \sum_{k=m}^{n} a_k \right| < \varepsilon.
\]

Also, note that by the Cesaro-Stolz's theorem, we have that

\[
\lim_{n \to \infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = \lim_{n \to \infty} a_{n+1} = 0.
\]

This implies that \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=m}^{n} a_k = 0\), for all \(m \in \mathbb{N}\). Since the sum of the \(m-1\) first terms is fixed and finite, taking \(m = 1, 2, \ldots, N_1\) and since \(N_1\) is fixed and depending on \(\varepsilon\), we can find \(N_2 \in \mathbb{N}\), with \(N_2 > N_1\) such that for all \(n \geq N_2 > N_1\), we have

\[
(*) \quad \left| \sum_{k=1}^{n} a_k \right| < \frac{\varepsilon}{N_1}.
\]
for all $j = 1, 2, \ldots, N_1$. Now, let us consider $\frac{1}{n} \sum_{k=1}^{n} k a_k$ for $n \geq N_2$. We can rewrite the sum as

$$\frac{1}{n} \sum_{k=1}^{n} k a_k = \frac{1}{n} \sum_{k=1}^{n} a_k + \frac{1}{n} \sum_{k=2}^{n} a_k + \cdots + \frac{1}{n} \sum_{k=N_1}^{n} a_k + \frac{1}{n} \sum_{k=N_1+1}^{n} a_k + \frac{1}{n} \sum_{k=n}^{n} a_k.$$  

By the triangle inequality, (*) we obtain

$$\left| \frac{1}{n} \sum_{k=1}^{n} k a_k \right| \leq \frac{N_1}{N_1} \varepsilon + \frac{n-N_1}{n} \varepsilon \leq 2 \varepsilon$$

and since $\varepsilon$ was arbitrary, we are done. \(\square\)
Problem. (OSU Qualifying 2003 & 2007)

Let \((a_n)_{n=1}^\infty\) be a sequence of positive reals such that the series \(\sum a_n\) is convergent. Show that the sequence \(\sum \frac{1}{n^2}\) converges also.

**Solution 1.**

By AM-GM inequality, we have:

\[
\frac{a_n + \frac{1}{n^2}}{2} \geq \sqrt{a_n \cdot \frac{1}{n^2}} = \frac{\sqrt{a_n}}{n}.
\]

Now, since the series \(\sum a_n\) and \(\sum \frac{1}{n^2}\) are convergent, it follows by the comparison test, it follows that the series \(\sum \frac{\sqrt{a_n}}{n}\) is convergent and thus the conclusion follows.

**Solution 2.**

By the Cauchy-Schwarz inequality, we have:

\[
\left(\sum_{k=1}^n \frac{\sqrt{a_k}}{k}\right)^2 \leq \sum_{k=1}^n (\sqrt{a_k})^2 \cdot \left(\sum_{k=1}^n \frac{1}{k^2}\right) = \left(\sum_{k=1}^n a_k\right) \cdot \left(\sum_{k=1}^n \frac{1}{k^2}\right).
\]

\[
\Rightarrow \sum_{n=1}^\infty \frac{\sqrt{a_n}}{n} \leq \sqrt{\sum_{n=1}^\infty a_n} \cdot \sqrt{\sum_{n=1}^\infty \frac{1}{n^2}} \to \infty \quad \text{as} \quad n \to \infty,
\]

\[\Rightarrow \text{convergent} \]

\[\Rightarrow \text{by the comparison test} \Rightarrow \text{conclusion}. \]

**Remark.**

\[a_n^3 + \frac{1}{n^2} + \frac{1}{n^2} \geq 3 \sqrt[3]{a_n^3 \cdot \frac{1}{n^2} \cdot \frac{1}{n^2}} = 3 \sqrt[3]{\frac{a_n}{n^2}}.\]

This means that if the series \(\sum a_n^3\) converges, then the series \(\sum \frac{a_n}{n^2}\) converges.
**Problem.** (University of California Davis Qualifying Exam, 2011)

Let \((a_n)_{n=1}^{\infty}\) be a sequence of real positive numbers such that \(\sum_{n=1}^{\infty} a_n^3\) converges. Show that \(\sum_{n=1}^{\infty} \frac{a_n}{n}\) converges as well.

**Solution.**

By Hölder's inequality, we have:

\[
\left( \sum_{k=1}^{m} a_k^3 \right)^{\frac{1}{3}} \left( \sum_{k=1}^{m} \frac{1}{k^{32}} \right)^{\frac{2}{3}} \geq \left( \sum_{k=1}^{m} \frac{a_k}{k} \right).
\]

By passing to the limit as \(m \to \infty\), we obtain:

\[
\left( \sum_{n=1}^{\infty} a_n^3 \right)^{\frac{1}{3}} \left( \sum_{n=1}^{\infty} \frac{1}{n^{32}} \right)^{\frac{2}{3}} \geq \left| \sum_{n=1}^{\infty} \frac{a_n}{n} \right|.
\]

Since the series \(\sum_{n=1}^{\infty} \frac{1}{n^{32}}\) converges, by the comparison test it follows that the series \(\sum_{n=1}^{\infty} \frac{a_n}{n}\) converges as well. \(\square\)
Problem. (University of Pittsburgh Preliminary Exam 2011)

Let \( \beta > 0 \) and \((u_n)_{n=1}^{\infty}\) be a sequence of positive reals such that

\[
\frac{u_n}{\ln u_n} \leq \beta
\]

for every positive integer \(n\). Prove that

\[
\limsup_{n \to \infty} \frac{u_n}{\ln u_n} \leq \limsup_{n \to \infty} \left( \frac{u_{n+1}}{u_n} \right).
\]

Solution.

Let us recall the Cesaro-Stolz's lemma:

\[
\limsup_{n \to \infty} \frac{a_n}{b_n} \leq \limsup_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}
\]

Indeed, we have:

\[
\sqrt{u_n} = \frac{u_n}{\ln u_n} = e^{\frac{\ln u_n}{\ln 2}} = \frac{\ln u_n}{\ln n}. \quad \text{By the Cesaro-Stolz's lemma, we have:}
\]

\[
\limsup_{n \to \infty} \frac{\ln u_n}{n} \leq \limsup_{n \to \infty} \frac{\ln (u_{n+1}) - \ln u_n}{n+1 - n}
\]

\[
= \limsup_{n \to \infty} \frac{\ln \left( \frac{u_{n+1}}{u_n} \right)}{n}
\]

This implies that:

\[
\limsup_{n \to \infty} \sqrt{u_n} = e \limsup_{n \to \infty} \frac{\ln u_n}{\ln n} \leq \limsup_{n \to \infty} \frac{u_{n+1}}{u_n}
\]

and we are done. \(\square\)
Problem. (OSU Qualifying Exam)

Show that the Bertrand sequence $(B_n)_{n \geq 2}$

$$B_n = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \cdots + \frac{1}{n\ln n}, n \geq 2$$

diverges.

Solution. We have:

$$B_{n+1} - B_n = \frac{1}{(n+1)\ln(n+1)} > 0 \implies B_n \text{ is strictly increasing.}$$

Suppose by contradiction that $B_n$ is bounded, so $B_n$ is convergent. Let us consider the function:

$$f(x) = \ln x - 1/x, \quad f: [1, \infty) \to \mathbb{R}. \text{ Clearly } f(x) = -\frac{1}{x\ln x},$$

so by the Lagrange's mean value theorem, there exists $c_k \in (k, k+1)$ such that

$$\frac{f(k+1) - f(k)}{(k+1) - k} = f'(c_k),$$

i.e.

$$\ln\ln(k+1) - \ln\ln k = \frac{1}{ck\ln c_k}.$$

But $c_k \in [k, k+1)$, so this implies that

$$\frac{1}{(k+1)\ln(k+1)} < \ln\ln(k+1) - \ln\ln k < \frac{1}{k\ln k} \quad \text{ for } k \geq 2.$$ 

So by adding up all these inequalities, we get

$$B_n > \sum_{k=2}^n \left(\ln\ln(k+1) - \ln\ln k\right) = \ln(\ln(n+1)) - \ln\ln 2 \to \infty,$$

so $B_n$ diverges. $\Box$
Problem. (Raabe-Duhamel)

(a) Let \((a_n)_{n=1}^{\infty}\) be a sequence of positive reals, and consider

\[ R_n = n \left( \frac{a_n}{a_{n+1}} - 1 \right), \quad n \geq 1. \]

If there exists \(\lim_{n \to \infty} R_n = L\), then show that:

(i) If \(L > 1\), then the series \(\sum_{n=1}^{\infty} a_n\) converges;

(ii) If \(L < 1\), then the series \(\sum_{n=1}^{\infty} a_n\) diverges;

(iii) If \(L = 1\), we cannot conclude upon the nature of the series \(\sum_{n=1}^{\infty} a_n\).

Solution.

(i) Assume that \(L > 1\) and let \(L > \alpha > 1\). Since

\[ \lim_{n \to \infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right) = L > \alpha, \]

there exists an index \(N\) such that

\[ n \left( \frac{a_n}{a_{n+1}} - 1 \right) > \alpha, \quad (\forall n \geq N). \]

This is equivalent to

\[ n a_n - (n+1) a_{n+1} > (\alpha - 1) a_{n+1}, \quad (\forall n \geq N) \]

\(\Rightarrow\) \(a_n\) is decreasing for all \(n \geq N\). By the monotone convergence theorem, it implies that \(a_n\) converges. This implies that the series \(\sum_{n=1}^{\infty} (n a_n - (n+1) a_{n+1})\) converges, so by the comparison test, \(\sum_{n=1}^{\infty} a_n\) converges.
(ii) Now, assume that $L < 1$. It follows that the sequence $(a_n)_{n \in \mathbb{N}}$ is increasing, so again, we can apply the comparison test.\[\square\]

Because $a_n > (n \cdot a_n) \frac{1}{n}$ and we are done.

**Application.**

Study the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(\frac{3}{2})^n}$.

We have

\[
n \left( \frac{a_n}{a_{n+1}} - 1 \right) = n \left[ \frac{\frac{e}{(1+1)^n}}{(1+1)^n - 1} \right].
\]

Since $\lim_{x \to 0} \frac{e}{(1+x)^x} = \frac{1}{2}$, the series diverges.