This exam contains 12 pages (including this cover page) and 11 questions. Total of points is 33. The last problem is for BONUS points.

This is NOT an open book and notes exam. No calculators are allowed. Show all your work (no work = no credit). Write neatly. Simplify your answers.

Grade Table (for teacher use only)

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Good luck !!!

Good job!
You really know Calculus III!
1. (3 points) For what values of \( a \) are the vectors \((a - 1, 2)\) and \((a - 4, 1)\) orthogonal?

The dot product must be zero:

\[
\begin{align*}
0 &= (a-1, 2) \cdot (a-4, 1) = (a-1)(a-4) + 2, \\
0 &= a^2 - 4a - a + 4 + 2 = a^2 - 5a + 6 = (a-2)(a-3)
\end{align*}
\]

This implies \[a = 2\] and \[a = 3\]
2. (3 points) Find the angle between the planes \( x + 2 = y - z \) and \( 2x - y = z \).

The planes are \[
\begin{cases}
  x - y + 2z + 2 = 0 \\
  2x - y - z = 0
\end{cases}
\]

The normal vectors for both planes are given by

\[ \mathbf{n}_1 = (1, -1, 1) \quad \text{and} \quad \mathbf{n}_2 = (2, -1, -1). \]

Therefore, we have

\[
\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|} = \frac{(1, -1, 1) \cdot (2, -1, -1)}{\sqrt{3} \cdot \sqrt{6}} = \frac{2 + 1 - 1}{3 \sqrt{2}} = \frac{2}{3 \sqrt{2}}.
\]

This implies that

\[ \theta = \arccos \left( \frac{2}{3 \sqrt{2}} \right) \]
3. (3 points) Find the equation of a plane passing through the points $A(1,1,1), B(2,2,2)$ and $C(1,2,3)$.

We have

\[ \overrightarrow{AB} = (2-1, 2-1, 2-1) = (1, 1, 1) \]
\[ \overrightarrow{AC} = (1-1, 2-1, 3-1) = (0, 1, 2) \]

This implies that

\[ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (1, -2, 1) - \text{normal} \]

The plane passing through the point $(1,1,1)$ with the normal $(1,-2,1)$ is

\[ 1 \cdot (x-1) + (-2) \cdot (y-1) + 1 \cdot (z-1) = 0 \]

which is equivalent to

\[ x - 2y + 2z - 4 = 0 \]

or

\[ x - 2y + z = 0 \]
4. (3 points) Find the area of the triangle with vertices \(A(1, 1, 1), B(2, 2, 2)\) and \(C(1, 2, 3)\).

The area of the triangle \(ABC\) is given by

\[
\frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{1^2 + (-2)^2 + 1^2} = \frac{1}{2} \sqrt{5}.
\]
5. (3 points) Classify the surface \( x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0 \) (ellipsoid, paraboloid, etc)

We have

\[
(x^2 - 8x + 16) + (y^2 - 6y + 9) + (z^2 - 8z + 16) = 17
\]

or equivalently

\[
(x-4)^2 + (y-3)^2 + (z-4)^2 = 17.
\]

This is a sphere of radius \( \sqrt{17} \) centered at point \( (4,3,4) \).
6. (3 points) Find the length of the curve \( r(x) = (x, f(x)) \), \( a \leq x \leq b \) where \( f \) is a given function.

We have
\[
\sqrt{x'(x)} = \sqrt{1 + (f'(x))^2} \quad \text{and thus}
\]
\[
|\sqrt{x'(x)}| = \sqrt{1 + (f'(x))^2}, \quad \text{so the length of the curve } r(x) \text{ is given by}
\]
\[
\int_a^b \sqrt{1 + (f'(x))^2} \, dx.
\]
7. (3 points) Show that the limit \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \) does not exist.

**First approach:**

We have \((x,y) = (x,0) \to (0,0) \Rightarrow \frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2}{x^2} = 1 \to 1\)

\((x,y) = (0,y) \to (0,0) \Rightarrow \frac{x^2 - y^2}{x^2 + y^2} = \frac{-y^2}{y^2} = -1 \to -1\).

Since we obtained different limits along the x-axis and along the y-axis, it follows that the limit \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \) does NOT exist.

**Second approach:**

Consider the line \( y = mx \), \( m \in \mathbb{R} \). Then, we have

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{(x,mx) \to (0,0)} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{(x,mx) \to (0,0)} \frac{x^2(1 - m^2)}{x^2(1 + m^2)} = \frac{1 - m^2}{1 + m^2},
\]

which depends on the parameter (real) \( m \).

Therefore, the limit does NOT exist.
8. (3 points) Compute the following limit: \( \lim_{(x,y) \to (0,0)} \frac{\log(1 + x^3 + y^2)}{x^3 + y^2} \).

It is clear that if \( (x,y) \to (0,0) \) then \( x^3 + y^2 \to 0 \). Denote \( x^3 + y^2 = \mu \to 0 \). Our limit transforms into

\[
\lim_{(x,y) \to (0,0)} \frac{\log(1 + x^3 + y^2)}{x^3 + y^2} = \lim_{\mu \to 0} \frac{\log(1 + \mu)}{\mu}
\]

Now, by applying L'Hospital rule, we get

\[
\lim_{\mu \to 0} \frac{\log(1 + \mu)}{\mu} = \lim_{\mu \to 0} \left( \frac{\log(1 + \mu)'}{\mu'} \right) = \lim_{\mu \to 0} \frac{1}{1 + \mu} = 1
\]

In conclusion, our limit equals 1.
9. (3 points) Compute the following limit: \[ \lim_{(x,y) \to (0,0)} \frac{x^3y + x^2y^2}{x^2 + y^2}. \]

We have that our limit is the same as 
\[ \lim_{(x,y) \to (0,0)} \frac{xy(x+y)}{x^2 + y^2}. \]

We have:

\[
0 \leq \left| \frac{x^2y(x+y)}{x^2 + y^2} \right| = \left( \left| \frac{x^2}{x^2 + y^2} \right| \left| y(x+y) \right| \right) \leq \\
\leq 1 \cdot \left| y(x+y) \right| \xrightarrow{(x,y) \to (0,0)} 0.
\]

Therefore, by the squeeze theorem, it follows that 
\[ \lim_{(x,y) \to (0,0)} \left| \frac{x^2y(x+y)}{x^2 + y^2} \right| = 0. \]
10. (3 points) For what values of \( a \) is the function

\[
f(x, y) = \begin{cases} 
\frac{x^2 + y^2 + x^2 \sin(xy)}{a} & (x, y) \neq (0, 0) \\
\frac{x^2}{x^2 + y^2} & (x, y) = (0, 0)
\end{cases}
\]

continuous?

We have

\[
f(x, y) = \frac{x^2 + y^2}{x^2 + y^2} + \frac{x^2 \sin(xy)}{x^2 + y^2}
\]

\[
= 1 + \frac{x^2 \sin(xy)}{x^2 + y^2}
\]

It follows that

\[
\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \left(1 + \frac{x^2 \sin(xy)}{x^2 + y^2}\right) = 1 + \lim_{(x, y) \to (0, 0)} \frac{x^2 \sin(xy)}{x^2 + y^2}
\]

On the other hand,

\[
0 \leq \left| \frac{x^2 \sin(xy)}{x^2 + y^2} \right| = \left| \left( \left| \frac{x^2}{x^2 + y^2} \right| \right) \sin(xy) \right| < \left| \sin(xy) \right|
\]

So, \( \lim_{(x, y) \to (0, 0)} f(x, y) = 1 + 0 = 1 \). Since \( f \) is continuous at \((0, 0)\), we have \( \lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0) = a \), so \(|a| = 1\).
11. (3 points) (BONUS!) Let \( \vec{u} = \text{proj}_a \vec{b} \) and \( \vec{v} = \text{proj}_b \vec{a} \). Show that \( \frac{|\vec{u} \times \vec{v}|}{\vec{u} \cdot \vec{v}} = \frac{|\vec{a} \times \vec{b}|}{\vec{a} \cdot \vec{b}} \).

We have

\[
|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta = \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} |\vec{a}| \cdot \frac{\vec{b} \cdot \vec{a}}{|\vec{b}|^2} \sin \theta
\]

\[
= \frac{|\vec{a}| |\vec{b}| \cos^2 \theta \sin \theta}{|\vec{a}| |\vec{b}| \cos^2 \theta} = |\vec{a}| |\vec{b}| \cos \theta.
\]

On the other hand,

\[
|\vec{u} \cdot \vec{v}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} |\vec{a}| \cdot \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} |\vec{b}| = |\vec{a}| |\vec{b}| \cos \theta
\]

Therefore, we have:

\[
\frac{|\vec{u} \times \vec{v}|}{|\vec{u} \cdot \vec{v}|} = \frac{|\vec{a}| |\vec{b}| \cos^2 \theta \sin \theta}{|\vec{a}| |\vec{b}| \cos^3 \theta} = \frac{|\vec{a}| |\vec{b}| \sin \theta}{|\vec{a}| |\vec{b}| \cos \theta} = \frac{|\vec{a} \times \vec{b}|}{\vec{a} \cdot \vec{b}}.
\]

Remark.

In fact, we have the following equality:

\[
\frac{|\vec{u} \times \vec{v}|}{\vec{u} \cdot \vec{v}} = \frac{|\vec{a} \times \vec{b}|}{\vec{a} \cdot \vec{b}} = \tan \theta.
\]