Advanced Calculus, Mid-term exam 3/17/16
Name:

20 points per question.
The best six questions will count.

Question 1
Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be given by the formulas:

\[
\begin{align*}
  f(0, 0) &= 0, \\
  f(x, y) &= \frac{(x - y)^3}{x^2 + y^2}, \quad \text{for any } (x, y) \in \mathbb{R}^2, \text{ with } (x, y) \neq (0, 0).
\end{align*}
\]

• Prove that \( f \) is everywhere Lipschitz, but not everywhere differentiable.

Question 2
Let \( f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (-1, 1) \to \mathbb{R} \) be given by the formulas:

\[
\begin{align*}
  f(x, \sin(x)) &= -1, \quad \text{for any } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\
  f(x, y) &= \frac{x - \arcsin(y)}{y - \sin(x)}, \quad \text{for any } (x, y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (-1, 1) \to \mathbb{R}, \text{ with } y \neq \sin(x).
\end{align*}
\]

• Where is \( f \) continuous and where is \( f \) differentiable? Discuss.

Question 3
Let \( f(x, y) = (x - y, xy) \), for any \((x, y)\) in \( \mathbb{R}^2 \), with \( x > 0 \) and \( y > 0 \).

• Show that \( f \) is bijective onto its image and has a smooth inverse.

• Also identify the image of \( f \) and determine the Jacobian matrix of the inverse function.
Question 4
Let $F = \{f_n : \mathbb{R} \to \mathbb{R}; n \in \mathbb{N}\}$ be a sequence of $C^1$ functions satisfying the conditions, for each $n \in \mathbb{N}$:

- $f_n(0) = 0$,
- $|f_n'(x)| < \frac{n^2 + x^4}{n^2 + x^2}$, for all $x \in \mathbb{R}$.

Show that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a subsequence $G = \{f_{n_k} : k \in \mathbb{N}\}$ of $F$, such that for each $x \in \mathbb{R}$:

$$\lim_{k \to \infty} f_{n_k}(x) = f(x).$$

Also prove that the convergence of the subsequence $G$ to its limit is uniform on compact subsets of the reals.

Question 5
For $n$ a positive integer, let $u : \mathbb{R}^n - \{0\} \to \mathbb{R}$ be a $C^2$ function. Suppose that $u(x)$ depends only on the variable $r = \sqrt{\sum x_i^2}$, for $x \in \mathbb{R}^n - \{0\}$ and that $u$ is bounded on its domain. Finally suppose also that $u$ is harmonic: $\nabla \cdot \nabla u = 0$.

- Prove that $u$ is constant.

Question 6
Let $A$ and $B$ be measurable subsets of the reals.

- Prove that:

$$m^*(A) + m^*(B) = m^*(A \cup B) + m^*(A \cap B).$$

- If we drop the condition that the set $A$ be measurable is this equation still true? Discuss.
- If we drop the condition that the sets $A$ and $B$ be measurable is this equation still true? Discuss.
Question 7
Let \( \{A_i : i \in \mathbb{N}\} \) be a sequence of subsets of the reals, such that:
\[
\lim_{n \to \infty} m^*(A_n) = 0.
\]

- Prove that the following set \( A \) is measurable:
\[
A = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.
\]

Question 8
Let \( X \) be a measurable subset of the reals, with finite positive measure \( m(X) \).
Let \( f : X \to \mathbb{R} \) be bounded and measurable, such that \( f \) obeys the relation:
\[
\left( \int_X f \right)^2 = m(X) \int_X f^2.
\]
Prove that \( f \) is almost everywhere constant.

Question 9
Let \( f : \mathbb{R} \to \mathbb{R} \) be \( C^1 \).
Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by the formula, defined for any \((x, y) \in \mathbb{R}^2\):
\[
F(x, y) = (f(x), xf(x) - y).
\]
Suppose \( a \in \mathbb{R} \) is such that \( f'(a) \neq 0 \).

- Prove that for any \( b \in \mathbb{R} \), \( F \) is invertible near the point \((a, b)\), and express the function \( F^{-1} \) in terms of the (local) inverse function of the function \( f \).

- If \( f \) is strictly monotonic, is \( F \) globally invertible? Discuss.
Question 10

Let $\mathbb{V}$ be a real vector space.
Let $h$ be a non-zero element of $\mathbb{V}^*$, the dual space of $\mathbb{V}$.
Let $g : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be a symmetric bilinear form, such that $g$ is positive definite:

$$g(x, x) \geq 0, \text{ for all } x \in \mathbb{V} \text{ and } g(x, x) = 0 \text{ if and only if } x = 0.$$ 

Put $P = g(x, x)$ and $Q = h(x)$, defined for $x \in \mathbb{V}$, so $P$ is a quadratic form on $\mathbb{V}$, whereas $Q$ is a linear form.

- Find the critical values, if any, of the function $Q$ restricted to the space $P = c$, for $c$ a given positive real constant and discuss whether or not these critical values form local maxima or minima.

- Find the critical values, if any, of the function $P$ restricted to the space $Q = b$, for $b$ given real constant and discuss whether or not these critical values form local maxima or minima.