WORKSHEET FOR THE PRELIMINARY EXAMINATION-REAL ANALYSIS (DIFFERENTIABILITY OF SINGLE VARIABLE FUNCTIONS)

INSTRUCTOR: CEZAR LUPU

**Problem 1.**

a) [Bernoulli] Show that for any $x > -1$, $x \neq 0$, and $\alpha > 0$ we have the following:

\[
\begin{cases}
(1 + x)^\alpha > 1 + \alpha x, & \text{if } \alpha > 1 \\
(1 + x)^\alpha = 1 + \alpha x, & \text{if } \alpha = 1 \\
(1 + x)^\alpha < 1 + \alpha x, & \text{if } \alpha < 1 
\end{cases}
\]

b) [Young] Let $a, b \geq 0$ and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that

\[
\frac{a^p}{p} + \frac{b^q}{q} \geq ab.
\]

**Problem 2.** Let $f : [-1, 1] \to \mathbb{R}$ be a function such that $f(-1) = f(1) = 0$, $f$ is twice continuously differentiable on $[-1, 1]$ and $|f''(x)| \leq 1$ for all $x \in (-1, 1)$. Show that $|f(x)| \leq \frac{1}{2}$ on $[-1, 1]$. Find a function such that $\max_{x \in [-1,1]} |f(x)| = \frac{1}{2}$.

University of Missouri-Columbia Qualifying Exam, 2012

**Problem 3.** Let $f$ be a $C^2$ function on the real line. Assume $f$ is bounded with bounded second derivative. Let

\[
A = \sup_{x \in \mathbb{R}} |f(x)|, \quad B = \sup_{x \in \mathbb{R}} |f''(x)|.
\]

Prove that

\[
\sup_{x \in \mathbb{R}} |f'(x)| \leq 2\sqrt{AB}.
\]

University of Pittsburgh Preliminary Examination, 2008

**Problem 4.** (a) If $f$ is a $C^2$ function on an open interval, prove that

\[
\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x).
\]

Berkeley Preliminary Examination, 1996
(b) Suppose that \( f(x) \) is defined on an open interval containing \( x \), and \( f(x) \) is three times differentiable on this interval. Show that

\[
\lim_{h \to 0} \frac{f(x + h) - 3f(x) + 3f(x - h) - 2f(x - 2h)}{h^3} = f'''(x).
\]

University of New Mexico Qualifying Exam, 2001

**Problem 5.** Let \( y : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function that satisfies the differential equation

\[
y'' + y' - y = 0
\]

for \( x \in [0, L] \), where \( L \) is a positive real number. Suppose that \( y(0) = y(L) = 0 \). Prove that \( y \equiv 0 \) on \([0, L]\).

Berkeley Preliminary Exam, 1990

**Problem 6.** [Gronwall-Bellman] Let \( K \) be a real constant. Suppose that \( y(t) \) is a positive differential function satisfying \( y'(t) \leq K \cdot y(t) \), for \( t \geq 0 \). Prove that \( y(t) \leq e^{Kt} y(0) \) for \( t \geq 0 \).

Berkeley Preliminary Exam, 1998

**Problem 7.** a) Suppose \( f \) is a differentiable function from reals into reals. Suppose \( f'(x) > f(x) \) for all \( x \in \mathbb{R} \), and \( f(x_0) = 0 \). Prove that \( f(x) > 0 \) for all \( x > x_0 \).

b) Suppose \( f \) is twice differentiable real-valued function on \( \mathbb{R} \) such that \( f(0) = 0 \), \( f'(0) > 0 \), and \( f''(x) \geq f(x) \), for all \( x \geq 0 \). Prove that \( f(x) > 0 \) for all \( x > 0 \).

University of Missouri-Columbia Qualifying Exam, 2014

**Problem 8.** Let \( f : [0, 1] \to \mathbb{R} \) be continuous function with \( f(0) = f(1) = 0 \). Assume that \( f'' \) exists on \( 0 < x < 1 \) and \( f'' + 2f' + f \geq 0 \). Show that \( f(x) \leq 0 \) for all \( 0 \leq x \leq 1 \).

Berkeley Preliminary Exam, 1984

**Problem 9.** Let \( f \) be a positive function of class \( C^2 \) on \((0, \infty)\) such that \( f' \leq 0 \) and \( f'' \) is bounded. Prove that \( \lim_{t \to \infty} f'(t) = 0 \).

Berkeley Preliminary Exam, 2000

**Problem 10.** Let \( I \subset \mathbb{R} \) be an open interval and let \( f, g : I \to \mathbb{R} \) be differentiable functions. Prove that between two consecutive zeros of \( f \) there is at least one zero of \( f' + gf' \).

**Problem 11.** Let \( f \) be an infinitely differentiable function from \( \mathbb{R} \) to \( \mathbb{R} \). Suppose that, for some positive integer \( n \),
\[ f(1) = f(0) = f'(0) = f'' = \ldots = f^{(n)}(0) = 0. \]
Prove that \( f^{(n+1)}(x) = 0 \) for some \( x \in (0, 1) \).

Berkeley Preliminary Exam, 1991

**Problem 12.** Let \( f \) be a continuous real-valued function on \([0, 1]\) such that, for each \( x_0 \in [0, 1) \)
\[
\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.
\]
Prove that \( f \) is nondecreasing.

**Problem 13.**

(a) Let \( I \) be an open interval in \( \mathbb{R} \). Show that a convex function \( f : I \to \mathbb{R} \) is continuous.

(b) Let \( f : I \to \mathbb{R} \) be a continuous and satisfy the inequality
\[
f \left( \frac{x + y}{2} \right) \leq \frac{1}{2}(f(x) + f(y)),
\]
for all \( x, y \in I \), then \( f \) is convex.

Ohio State University Qualifying Exam, 2012

**Problem 14.** Let \( f, g \) and \( h \) be real-valued functions which are continuous on \([a, b]\) and differentiable on \((a, b)\), where \( a, b \in \mathbb{R} \) with \( a < b \). Define \( F \) on \([a, b]\) by
\[
F(x) = \begin{bmatrix}
  f(x) & g(x) & h(x) \\
  f(a) & g(a) & h(a) \\
  f(b) & g(b) & h(b)
\end{bmatrix}
\]
Show that there exists \( c \in (a, b) \) such that \( F'(c) = 0 \).

Ohio State University Qualifying Exam, 2011

**Problem 15.** Let \( f \) be a differentiable function on \( \mathbb{R} \). Assume that there is no \( x \) such that \( f(x) = 0 = f'(x) \). Show that the set
\[
S = \{ x \in [0, 1] : f(x) = 0 \}
\]
is finite.

University of Pittsburgh Preliminary Exam, 2005 & 2008

**Problem 16.**

(i) Assume that \( f \in C^2(\mathbb{R}) \) satisfies \( f''(x) \geq 0 \) for all \( x \in \mathbb{R} \) and \( f'(0) = 0 \). Prove that \( f(x) \geq f(0) \) for each \( x \in \mathbb{R} \).

(ii) Let \( a_1, a_2, \ldots, a_n > 0 \) be \( n \) given numbers. Prove that the following are equivalent:

- \( a_1^x + a_2^x + \ldots + a_n^x \geq n \), for each \( x \in \mathbb{R} \);
- \( a_1 a_2 \ldots a_n = 1 \).
(iii) Deduce the arithmetic-geometric mean inequality:
\[
\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1a_2\ldots a_n}.
\]

University of Missouri-Columbia Qualifying Exam, 2002

**Problem 17.** (i) Let \( f : (0, \infty) \rightarrow \mathbb{R} \) be differentiable such that

\[
\lim_{x \to \infty} (f(x) + f'(x)) = 0.
\]

Show that \( \lim_{x \to \infty} f(x) = 0 \).

(ii) Let \( a \in \mathbb{R} \) and \( f : (a, \infty) \rightarrow \mathbb{R} \) is differentiable. Show that if there exists the limits

\[
\lim_{x \to \infty} f(x) = L_1 < \infty, \quad \lim_{x \to \infty} f'(x) = L_2 < \infty,
\]

then \( \lim_{x \to \infty} f'(x) = 0 \).

(iii) Let \( f : (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function with \( f' \) uniformly continuous. Show that if there exists \( \lim_{x \to \infty} f(x) \in \mathbb{R} \), then \( \lim_{x \to \infty} f'(x) = 0 \).

**Problem 18.** Suppose \( f(x) \) is infinitely differentiable on \( \mathbb{R} \) and \( f(a) = 0 \). Prove that \( f(x) = (x-a)g(x) \), where \( g(x) \) is also infinitely differentiable.

University of Pittsburgh Preliminary Examination, 2008

**Problem 19.** Let \( a < b \) and \( f \in C^2([a,b]) \). Suppose that \( f'(a) = f'(b) = 0 \). Prove that

\[
|f(b) - f(a)| \leq \left( \frac{b-a}{2} \right)^2 \sup_{x \in [a,b]} |f''(x)|.
\]

**Problem 20.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be twice continuously differentiable, \( a,b \in \mathbb{R} \) and \( a < b \). Suppose that \( f(a) = f(b) = 0 \) and \( |f''(x)| \leq 1 \) for every \( x \in [a,b] \). Prove that

\[
|f(x)| \leq \frac{(b-a)^2}{8},
\]

for every \( x \in [a,b] \).

University of Pittsburgh Preliminary Exam, 2013

**Problem 21.** Suppose that \( f : [0, \infty) \rightarrow \mathbb{R} \) is continuous on \([0, \infty)\) and differentiable on \((0, \infty)\), \( f(0) = 0 \), and \( \lim_{x \to \infty} f(x) = 0 \). Prove that there exists a point \( c \in (0, \infty) \) such that \( f'(c) = 0 \).

University of Pittsburgh Preliminary Exam, 2005

**Problem 22.** Show that if \( a_0, a_1, \ldots, a_n \) are real constants and
\[ a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0, \]

then the equation \( a_0 + a_1 x + \ldots + a_n x^n = 0 \) has at least one real root between 0 and 1.

University of Missouri-Columbia Qualifying Exam, 1994

Problem 23. Let \( f : [1, \infty) \to \mathbb{R} \) be a differentiable function on \([1, \infty)\). Suppose that \( \lim_{x \to \infty} \frac{f(x)}{x} = 1 \). Prove that there exists a sequence \((x_n)_{n \geq 1}\) such that \( \lim_{n \to \infty} x_n = \infty \) and \( \lim_{n \to \infty} f'(x_n) = 1 \).

Problem 24. Let \( f : (0, 1] \to \mathbb{R} \) be differentiable with \( f(0) = 0 \) and there exists \( M > 0 \) such that

\[ |f'(x)| \leq M|f(x)|, \]

for all \( x \in [0,1] \). Show that \( f \equiv 0 \).

University of Missouri-Columbia Preliminary Exam, 2004

Problem 25. Consider the function \( f : \mathbb{R} \to \mathbb{R} \) given by

\[ f(x) = a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx, \]

where \( a_1, a_2, \ldots, a_n \) are reals, and \( n \in \mathbb{N} \). If \( |f(x)| \leq |\sin x| \), for all reals \( x \), then show that

\[ |a_1 + 2a_2 + \ldots + na_n| \leq 1. \]

Problem 26. Let \( f : [0, \infty) \to \mathbb{R} \) be differentiable with \( f' \) continuous, \( f(0) = 1 \), and \( |f(x)| \leq e^{-x} \) for all \( x > 0 \). Show that there exists \( x_0 \in \mathbb{R} \) such that \( f'(x_0) = -e^{-x_0} \).

Problem 27. Let \( f \) be a real-valued function defined for all \( x \geq 1 \), satisfying \( f(1) = 1 \), and

\[ f'(x) = \frac{1}{x^2 + f^2(x)}. \]

Prove that \( \lim_{x \to \infty} f(x) \) exists and it is less than \( 1 + \frac{\pi}{4} \).

University of Pittsburgh Preliminary Examination, 2011

Problem 28. [Hardy-Littlewood] Let \( f : [0, \infty) \to \mathbb{R} \) be twice differentiable function such that \( \lim_{x \to \infty} f(x) = 0 \) and \( f'' \) is bounded. Prove that \( \lim_{x \to \infty} f'(x) = 0 \).

Problem 29. [Joric] Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( f^2 \) and \( f^3 \) are differentiable. Is \( f \) differentiable?
Problem 30. Does there exist a twice differentiable function $f: \mathbb{R} \to \mathbb{R}$ such that $f \geq 0$ and $f'' < 0$?

Problem 31. Suppose $h: \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $h(0) = 1$. Also, assume that $g: \mathbb{R} \to \mathbb{R}$ is continuous, satisfies $g(0) = 0$, and is differentiable on $(-\infty, 0) \cup (0, \infty)$. Furthermore, suppose that there exist constants $C > 0$ and $\epsilon > 0$ such that

$$|g'(x)| \leq \frac{C}{|x|^{1-\epsilon}},$$

for each $x \neq 0$. Finally, define

$$f(x) := xh(g(x)), x \in \mathbb{R}.$$

(a) Prove that $f$ is differentiable at $x = 0$ and determine $f'(0)$.
(b) Prove that $f'(x)$ exists at every point $x \in \mathbb{R}$ and that $f': \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$.

University of Missouri-Columbia Qualifying Exam, 2004

Problem 32. [Littlewood] If a function $f: [0, \infty) \to \mathbb{R}$ is $n + 1$ times differentiable, $\lim_{x \to \infty} f(x)$ is finite, and $f^{(n+1)}$ is bounded, then $\lim_{x \to \infty} f^{(n)}(x) = 0$.

Problem 33. [Hardy-Littlewood] If $f$ is twice differentiable on $[0, \infty)$, $f(x) = o(1)$, $f''(x) = O(1)$, for $x \to \infty$, then $f'(x) = o(1)$.

Problem 34. Let $f: I \to \mathbb{R}$ ($I$ is an interval of $\mathbb{R}$) be such that $f(x) > 0$, $x \in I$. Suppose that $e^{cx}f(x)$ is convex in $I$ for every real $c$. Show that $\log f(x)$ is convex in $I$.

Berkeley Preliminary Exam, 1982

Problem 35. Let $n \geq 3$. Consider an $n$-times continuously differentiable function $f \in C^n(\mathbb{R})$ such that $f^{(k)}(0) = 0$, for $k = 2, 3, \ldots n - 1$, and $f^{(n)}(0) = 0$. Clearly, by the mean value theorem for any $h > 0$ there is $0 < \theta(h) < h$ such that

$$f(h) - f(0) = hf'(\theta(h)).$$

Prove that

$$\lim_{h \to 0} \frac{\theta(h)}{h} = \frac{1}{n} \left(\frac{1}{n}\right)^{-\frac{1}{n-1}}.$$

University of Pittsburgh Preliminary Exam, 2015