WORKSHEET FOR THE PRELIMINARY EXAMINATION-REAL ANALYSIS (DIFFERENTIABILITY OF SINGLE VARIABLE FUNCTIONS)

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Problem 1. a) [Bernoulli] Show that for any $x > -1$, $x \neq 0$, and $\alpha > 0$ we have the following:

$$\begin{cases} (1+x)^\alpha > 1 + \alpha x, & \text{if } \alpha > 1 \\ (1+x)^\alpha = 1 + \alpha x, & \text{if } \alpha = 1 \\ (1+x)^\alpha < 1 + \alpha x, & \text{if } \alpha < 1 \end{cases}$$

b) [Young] Let $a, b \geq 0$ and $p, q > 0$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Show that

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

Problem 2. Let $f : [-1, 1] \to \mathbb{R}$ be a function such that $f(-1) = f(1) = 0$, f is twice continuously differentiable on $[-1, 1]$ and $|f''(x)| \leq 1$ for all $x \in (-1, 1)$. Show that $|f(x)| \leq \frac{1}{2}$ on $[-1, 1]$. Find a function such that $\max_{x\in[-1,1]} |f(x)| = \frac{1}{2}$.

University of Missouri-Columbia Qualifying Exam, 2012

Problem 3. Let f be a C^2 function on the real line. Assume f is bounded with bounded second derivative. Let

$$A = \sup_{x\in\mathbb{R}} |f(x)|, B = \sup_{x\in\mathbb{R}} |f''(x)|.$$

Prove that

$$\sup_{x\in\mathbb{R}} |f'(x)| \leq 2\sqrt{AB}.$$

University of Pittsburgh Preliminary Examination, 2008

Problem 4. (a) If f is a C^2 function on an open interval, prove that

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

Berkeley Preliminary Examination, 1996

(b) Suppose that $f(x)$ is defined on an open interval containing x, and $f(x)$ is three times differentiable on this interval. Show that

$$\lim_{h\to 0} \frac{f(x+h) - 3f(x) + 3f(x-h) - 2f(x-2h)}{h^3} = f'''(x).$$

University of New Mexico Qualifying Exam, 2001

Problem 5. Let $y : \mathbb{R} \to \mathbb{R}$ be a C^∞ function that satisfies the differential equation

$$y'' + y' - y = 0$$

for $x \in [0, L]$, where L is a positive real number. Suppose that $y(0) = y(L) = 0$. Prove that $y \equiv 0$ on $[0, L]$.

Berkeley Preliminary Exam, 1990

Problem 6. [Gronwall-Bellman] Let K be a real constant. Suppose that $y(t)$ is a positive differential function satisfying $y'(t) \leq K \cdot y(t)$, for $t \geq 0$. Prove that $y(t) \leq e^{Kt} y(0)$ for $t \geq 0$.

Berkeley Preliminary Exam, 1998

Problem 7. a) Suppose f is a differentiable function from reals into reals. Suppose $f'(x) > f(x)$ for all $x \in \mathbb{R}$, and $f(x_0) = 0$. Prove that $f(x) > 0$ for all $x > x_0$.

b) Suppose f is twice differentiable real-valued function on $\mathbb{R}$ such that $f(0) = 0$, $f'(0) > 0$, and $f''(x) \geq f(x)$, for all $x \geq 0$. Prove that $f(x) > 0$ for all $x > 0$.

University of Missouri-Columbia Qualifying Exam, 2014

Problem 8. Let $f : [0, 1] \to \mathbb{R}$ be continuous function with $f(0) = f(1) = 0$. Assume that f'' exists on $0 < x < 1$ and $f'' + 2f' + f \geq 0$. Show that $f(x) \leq 0$ for all $0 \leq x \leq 1$.

Berkeley Preliminary Exam, 1984

Problem 9. Let f be a positive function of class C^2 on $(0, \infty)$ such that $f' \leq 0$ and f'' is bounded. Prove that $\lim_{t\to\infty} f'(t) = 0$.

Berkeley Preliminary Exam, 2000

Problem 10. Let $I \subset \mathbb{R}$ be an open interval and let $f, g : I \to \mathbb{R}$ be differentiable functions. Prove that between two consecutive zeros of f there is at least one zero of $f' + fg'$.

Problem 11. Let f be an infinitely differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that, for some positive integer n,

$$f(1) = f(0) = f'(0) = f'' = \ldots = f^{(n)}(0) = 0.$$

Prove that $f^{(n+1)}(x) = 0$ for some $x \in (0, 1)$.

Berkeley Preliminary Exam, 1991

Problem 12. Let f be a continuous real-valued function on $[0, 1]$ such that, for each $x_0 \in [0, 1)$

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Prove that f is nondecreasing.

Problem 13. a) Let I be an open interval in $\mathbb{R}$. Show that a convex function $f : I \to \mathbb{R}$ is continuous.

b) Let $f : I \to \mathbb{R}$ be a continuous and satisfy the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)),$$

for all $x, y \in I$, then f is convex.

Ohio State University Qualifying Exam, 2012

Problem 14. Let f, g and h be real-valued functions which are continuous on $[a, b]$ and differentiable on (a, b), where $a, b \in \mathbb{R}$ with $a < b$. Define F on $[a, b]$ by

$$F(x) = \begin{bmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{bmatrix}$$

Show that there exists $c \in (a, b)$ such that $F'(c) = 0$.

Ohio State University Qualifying Exam, 2011

Problem 15. Let f be a differwntiable function on $\mathbb{R}$. Assume that there is no x such that $f(x) = 0 = f'(x)$. Show that the set

$$S = \{x \in [0, 1] : f(x) = 0\}$$

is finite.

University of Pittsburgh Preliminary Exam, 2005 & 2008

Problem 16. (i) Assume that $f \in C^2(\mathbb{R})$ satisfies $f''(x) \geq 0$ for all $x \in \mathbb{R}$ and $f'(0) = 0$. Prove that $f(x) \geq f(0)$ for each $x \in \mathbb{R}$.

(ii) Let $a_1, a_2, \ldots, a_n > 0$ be n given numbers. Prove that the following are equivalent:

- $a_1^x + a_2^x + \ldots + a_n^x \geq n$, for each $x \in \mathbb{R}$;
- $a_1 a_2 \ldots a_n = 1$.

(iii) Deduce the arithmetic-geometric mean inequality:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

University of Missouri-Columbia Qualifying Exam, 2002

Problem 17. (i) Let $f : (0, \infty) \to \mathbb{R}$ be differentiable such that

$$\lim_{x\to\infty} (f(x) + f'(x)) = 0.$$

Show that $\lim\limits_{x\to\infty} f(x) = 0$.

(ii) Let $a \in \mathbb{R}$ and $f : (a, \infty) \to \mathbb{R}$ is differentiable. Show that if there exists the limits

$$\lim_{x\to\infty} f(x) = L_1 < \infty, \lim_{x\to\infty} f'(x) = L_2 < \infty,$$

then $\lim\limits_{x\to\infty} f'(x) = 0$.

(iii) Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function with f' uniformly continuous. Show that if there exists $\lim_{x\to\infty} f(x) \in \mathbb{R}$, then $\lim\limits_{x\to\infty} f'(x) = 0$.

Problem 18. Suppose $f(x)$ is infniitely differentiable on $\mathbb{R}$ and $f(a) = 0$. Prove that $f(x) = (x - a)g(x)$, where $g(x)$ is also infinitely differentiable.

University of Pittsburgh Preliminary Examination, 2008

Problem 19. Let $a < b$ and $f \in C^2([a, b])$. Suppose that $f'(a) = f'(b) = 0$. Prove that

$$|f(b) - f(a)| \leq \left(\frac{b-a}{2}\right)^2 \sup_{x\in[a,b]} |f''(x)|.$$

Problem 20. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable, $a, b \in \mathbb{R}$ and $a < b$. Suppose that $f(a) = f(b) = 0$ and $|f''(x)| \leq 1$ for every $x \in [a, b]$. Prove that

$$|f(x)| \leq \frac{(b-a)^2}{8},$$

for every $x \in [a, b]$.

University of Pittsburgh Preliminary Exam, 2013

Problem 21. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, $f(0) = 0$, and $\lim_{x\to\infty} f(x) = 0$. Prove that there exists a point $c \in (0, \infty)$ such that $f'(c) = 0$.

University of Pittsburgh Preliminary Exam, 2005

Problem 22. Show that if $a_0, a_1, \ldots, a_n$ are real constants and

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0,$$

then the equation $a_0 + a_1 x + \ldots + a_n x^n = 0$ has at least one real root between 0 and 1.

University of Missouri-Columbia Qualifying Exam, 1994

Problem 23. Let $f : [1, \infty) \to \mathbb{R}$ be a differentiable function on $[1, \infty)$. Suppose that $\lim_{x\to\infty} \frac{f(x)}{x} = 1$. Prove that there exists a sequence $(x_n)_{n\geq 1}$ such that $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} f'(x_n) = 1$.

Problem 24. Let $f : (0, 1] \to \mathbb{R}$ be differentiable with $f(0) = 0$ and there exists $M > 0$ such that

$$|f'(x)| \leq M|f(x)|,$$

for all $x \in [0, 1]$. Show that $f \equiv 0$.

University of Missouri-Columbia Preliminary Exam, 2004

Problem 25. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx,$$

where $a_1, a_2, \ldots, a_n$ are reals, and $n \in \mathbb{N}$. If $|f(x)| \leq |\sin x|$, for all reals x, then show that

$$|a_1 + 2a_2 + \ldots + na_n| \leq 1.$$

Problem 26. Let $f : [0, \infty) \to \mathbb{R}$ be differntiable with f' continuous, $f(0) = 1$, and $|f(x)| \leq e^{-x}$ for all $x > 0$. Show that there exists $x_0 \in \mathbb{R}$ such that $f'(x_0) = -e^{-x_0}$.

Problem 27. Let f be a real-valued function defined for all $x \geq 1$, satisfying $f(1) = 1$, and

$$f'(x) = \frac{1}{x^2 + f^2(x)}.$$

Prove that $\lim_{x\to\infty} f(x)$ exists and it is less than $1 + \frac{\pi}{4}$.

University of Pittsburgh Preliminary Examination, 2011

Problem 28. [Hardy-Littlewood] Let $f : [0, \infty) \to \mathbb{R}$ be twice differentiable function such that $\lim_{x\to\infty} f(x) = 0$ and f'' is bounded. Prove that $\lim_{x\to\infty} f'(x) = 0$.

Problem 29. [Joric] Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f^2 and f^3 are differentiable. Is f differentiable?

Problem 30. Does there exist a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f \geq 0$ and $f'' < 0$?

Probem 31. Suppose $h : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $h(0) = 1$. Also, assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous, satisfies $g(0) = 0$, and is differentiable on $(-\infty, 0) \cup (0, \infty)$. Furthermore, suppose that there exist constants $C > 0$ and $\epsilon > 0$ such that

$$|g'(x)| \leq \frac{C}{|x|^{1-\epsilon}},$$

for each $x \neq 0$. Finally, define

$$f(x) := xh(g(x)), x \in \mathbb{R}.$$

(a) Prove that f is differentiable at $x = 0$ and determine $f'(0)$.
(b) Prove that $f'(x)$ exists at every point $x \in \mathbb{R}$ and that $f' : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$.

University of Missouri-Columbia Qualifying Exam, 2004

Problem 32. [Littlewood] If a function $f : [0, \infty) \to \mathbb{R}$ is $n + 1$ times differentiable, $\lim_{x\to\infty} f(x)$ is finite, and $f^{(n+1)}$ is bounded, then $\lim_{x\to\infty} f^{(n)}(x) = 0$.

Problem 33. [Hardy-Littlewood] If f is twice differentiable on $[0, \infty)$, $f(x) = o(1)$, $f''(x) = O(1)$, for $x \to \infty$, then $f'(x) = o(1)$.

Problem 34. Let $f : I \to \mathbb{R}$ (I is an interval of $\mathbb{R}$) be such that $f(x) > 0$, $x \in I$. Suppose that $e^{cx} f(x)$ is convex in I for every real c. Show that $\log f(x)$ is convex in I.

Berkeley Preliminary Exam, 1982

Problem 35. Let $n \geq 3$. Consider an n-times continuously differentiable function $f \in C^n(\mathbb{R})$ such that $f^{(k)}(0) = 0$, for $k = 2, 3, \ldots n - 1$, and $f^{(n)}(0) = 0$. Clearly, by the mean value theorem for any $h > 0$ there is $0 < \theta(h) < h$ such that

$$f(h) - f(0) = hf'(\theta(h)).$$

Prove that

$$\lim_{h\to 0} \frac{\theta(h)}{h} = \left(\frac{1}{n}\right)^{\frac{1}{n-1}}.$$

University of Pittsburgh Preliminary Exam, 2015