Problem 1. If \( f = (f_1, f_2, \ldots, f_n) : [a, b] \to \mathbb{R}^n \) is a continuous function, then we define
\[
\int_a^b f(t) dt = \left\langle \int_a^b f_1(t) dt, \int_a^b f_2(t) dt, \ldots, \int_a^b f_n(t) dt \right\rangle.
\]
Show that
\[
\left\| \int_a^b f(t) dt \right\| \leq \int_a^b \| f(t) \| dt.
\]

Problem 2. Let \( A = [a_{ij}]_{1 \leq i,j \leq n} \) be the matrix of a linear mapping \( A \in L(\mathbb{R}^n; \mathbb{R}^m) \). Prove that the operatorial norm,
\[
\| A \| = \sup_{\| x \| = 1} \| Ax \|
\]
satisfies the inequality
\[
\| A \| \leq \| A \|_{HS} := \left( \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2}
\]
Moreover, if \( A \) and \( B \) are \( n \times n \) matrices, prove that
\[
\| AB \|_{HS} \leq \| A \|_{HS} \| B \|_{HS}.
\]

Problem 3. Prove that to every \( A \in L(\mathbb{R}^n; \mathbb{R}^1) \) corresponds a unique \( y \in \mathbb{R}^n \) such that \( Ax = \langle x, y \rangle \). Prove also that \( \| A \| = \| y \| \).

Problem 4. Suppose \( f \) is differentiable mapping of \( \mathbb{R}^1 \) into \( \mathbb{R}^3 \) such that \( |f(t)| = 1 \) for every \( t \). Prove that \( f(t) \cdot f'(t) = 0 \).

Problem 5. If \( f \) is differentiable mapping of a connected open set \( E \subset \mathbb{R}^n \) into \( \mathbb{R}^m \), and if \( f'(x) = 0 \) for every \( x \in E \), prove that \( f \) is constant on \( E \).

Problem 6. Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable and \( F : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( F(x, y) = f(xy) \). Prove that
\[ \frac{\partial F}{\partial x} = y \frac{\partial F}{\partial y}. \]

**Problem 7.** Find all points \((x, y) \in \mathbb{R}^2\) where the function
\[ f(x, y) = |e^x - e^y|(x + y - 2) \]
is differentiable.

**Problem 8.**
(a) Prove that the function \(f : \mathbb{R}^2 \to \mathbb{R},\)
\[ f(x, y) = \begin{cases} 
1 - \cos((x + y)^2) & \text{if } (x, y) \neq (0, 0) \\
\frac{x^2 + y^2}{x^2 + y^2} & \text{if } (x, y) = (0, 0)
\end{cases} \]
is continuous.
(b) Prove that the function \(f : \mathbb{R}^2 \to \mathbb{R},\)
\[ f(x, y) = \begin{cases} 
x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases} \]
is continuous.

**Problem 9.** Define \(f : \mathbb{R}^2 \to \mathbb{R}\) by \(F(x, 0) = 0\) and
\[ f(x, y) = \left(1 - \cos \frac{x^2}{y^2}\right) \sqrt{x^2 + y^2}; \ y \neq 0. \]

(a) Show that \(f\) is continuous at \((0, 0)\).
(b) Calculate all the directional derivatives of \(f\) at \((0, 0)\).
(c) Show that \(f\) is not differentiable at \((0, 0)\).

**Problem 10.** If \(f(0, 0) = 0\) and
\[ f(x, y) = \frac{xy}{x^2 + y^2}, \]
if \((x, y) \neq (0, 0),\) prove that the first order partial derivatives \(\frac{\partial f}{\partial x}(x, y)\) and \(\frac{\partial f}{\partial y}(x, y)\) exist at every point of \(\mathbb{R}^2\), although is not continuous at \((0, 0)\).

**Problem 11.** Define \(f(0, 0) = 0\) and \(f(x, y) = \frac{x^3}{x^2 + y^2}, \text{ if } (x, y) \neq (0, 0)\).

(a) Prove that \(\frac{\partial f}{\partial x}(x, y)\) and \(\frac{\partial f}{\partial y}(x, y)\) are bounded functions in \(\mathbb{R}^2\). (Hence \(f\) is continuous)
(b) Let \(u\) be any unit vector in \(\mathbb{R}^2\). Show that the directional derivative \((D_u f)(0, 0)\) exists, and that its absolute value is at most 1.
**Problem 12.** Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by
\[
f(x, y) = \begin{cases} 
x^{4/3} \sin \left( \frac{y}{x} \right) & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]
Determine all points at which \( f \) is differentiable.

*Berkeley Preliminary Exam, 1986*

**Problem 13.** Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable and \( f \) satisfies \( f(tx) = t^nf(x) \) for all \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( t \in \mathbb{R} \). Prove that
\[
\sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}(x) = nf(x)
\]
for \( x \in \mathbb{R}^n \).

*University of Pittsburgh Preliminary Exam, 2008*

**Problem 14.** Suppose that \( f \in C^2(\mathbb{R}^n - \{0\}) \) depends on \( r = |x| \) only, i.e. \( f(x) = g(|x|) = g(r) \) for some \( g \in C^2((0, \infty)) \). Express the Laplace operator
\[
\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x)
\]
in terms of \( n, r, g \) and derivatives of \( g \) only.

*University of Pittsburgh Preliminary Examination, 2009*

**Problem 15.** Let \( \Omega \) be an open set in \( \mathbb{R}^n \). Let \( F : \Omega \to \mathbb{R}^n \) and \( G : \mathbb{R}^n \to \mathbb{R} \) be two continuously differentiable functions such that \( G \circ F = 0 \) on \( \Omega \). Suppose that
\[
\sum_{j=1}^{n} \left( \frac{\partial G(x)}{\partial x_j} \right)^2 > 0 \text{ for every } x \in F(\Omega).
\]
Prove that \( \det(DF) = 0 \) on \( \Omega \).

*University of Pittsburgh Preliminary Exam, 2012*

**Problem 16.** Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be given by the formulae:
\[
f(0, 0) = 0,
\]
\[
f(x, y) = \frac{(x+y)^3}{x^2 + y^2},
\]
for any \((x, y) \in \mathbb{R}^2\), with \((x, y) \neq (0, 0)\). Prove that \( f \) is everywhere Lipschitz, but not everywhere differentiable.

*University of Pittsburgh Preliminary Exam, 2014*

**Problem 17.** Decide which if the following functions are differentiable at \((0, 0)\):
(a) 
\[ f(x, y) = \begin{cases} \frac{x^3 y}{x^4 + y^2} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases} \]

(b) 
\[ f(x, y) = \begin{cases} \frac{x^2 y^2 - x^3}{x^2 + y^4} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases} \]

(c) 
\[ f(x, y) = \begin{cases} \frac{x^2(y^4 + 2x)}{x^2 + y^6} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases} \]

(d) 
\[ f(x, y) = \begin{cases} \frac{x^2 y^2}{x^4 + y^2} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases} \]

(e) 
\[ f(x, y) = \begin{cases} \frac{x^2 (x + y^2)}{x^2 + y^6} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases} \]

(f) 
\[ f(x, y) = \begin{cases} \frac{x^3 + y^3}{\sqrt{x^2 + y^2}} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases} \]

(g) 
\[ f(x, y) = \begin{cases} \frac{x^4 y^6 + x^3 + xy^4}{x^2 + y^4} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases} \]

(h) 
\[ f(x, y) = \begin{cases} \frac{|xy|^{3/2}}{x^2 + y^4} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases} \]

**Problem 18.** (a) Write down an example of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that the directional derivative \( f_u(0, 0) \) exists in \( \mathbb{R} \) for all unit vectors \( u \in \mathbb{R}^2 \), and yet \( f \) is not differentiable at \((0, 0)\). Also, prove these two facts for your example.

(b) Consider the function \( g : \mathbb{R}^2 \to \mathbb{R} \) given by

\[ g(x, y) = x^{2/3} y^{2/3} \]

for all \((x, y) \in \mathbb{R}^2\). Prove that \( g \) is differentiable at \((0, 0)\).
Problem 19. (a) Prove that if the partial derivatives (of first order) of a function \( f : \mathbb{R}^n \to \mathbb{R} \) exist everywhere and they are bounded, then \( f \) is continuous.

(b) Find a function \( f : \mathbb{R}^2 \to \mathbb{R} \) that is differentiable at each point, but whose partial derivatives are not continuous at \((0, 0)\).

Problem 20. Prove that for \( \alpha > 0 \) the mapping \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \), \( \Phi(x) = x|x|^\alpha \), i.e. \( \Phi(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi(x_n)) = (x_1|x|^\alpha, x_2|x|^\alpha, \ldots, x_n|x|^\alpha) \) is of class \( C^1 \) and find partial derivatives \( \frac{\partial \Phi_i}{\partial x_j} \).

Problem 21. Show that the vector field \( F(x) = x|x|^{-n} \) defined on \( \mathbb{R}^n - \{0\} \) is divergence free, i.e.
\[
\text{div} F(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{x_i}{|x|^n} \right) = 0,
\]
for all \( x \neq 0 \).

Problem 22. Consider on open ball \( B = B(a, r) \subset \mathbb{R}^n \). Prove that the function
\[
\phi(x) = \begin{cases} 
\exp \left( -\frac{1}{|x-a|^2 - r^2} \right) & \text{if } x \in B \\
0 & \text{if } x \in \mathbb{R}^n - B
\end{cases}
\]
is infinitely differentiable on \( \mathbb{R}^n \).

Problem 23. Let \( \alpha \in \mathbb{R} \) and \( f_\alpha : \mathbb{R}^2 \to \mathbb{R} \) be given by the formulae:
\[
f_\alpha(0, 0) = 0, \\
f_\alpha(x, y) = \frac{x^4 + y^4}{(x^2 + y^2)\alpha},
\]
for any \((x, y) \in \mathbb{R}^2 - \{(0, 0)\} \).
Determine with proof, those values of \( \alpha \) for which \( f_\alpha \) is differentiable.

University of Pittsburgh Preliminary Exam, 2014

Problem 24. Let \( f \) be a \( C^1 \) function from the interval \((-1, 1)\) into \( \mathbb{R}^2 \) such that \( f(0) = 0 \) and \( f'(0) \neq 0 \). Prove that there is a number \( \epsilon \in (0, 1) \) such that \( ||f(t)|| \) is an increasing function of \( t \) on \((0, \epsilon)\).

Berkeley Preliminary Exam, 1991

Problem 25. Let \( D \) be a non-empty, open and conex subset of \( \mathbb{R}^m \) and \( f : D \to \mathbb{R}^n \) is such that there exists \( \alpha > 1 \) and \( L > 0 \) with
\[
||f(x) - f(y)|| \leq L \cdot ||x - y||^\alpha,
\]
for all \( x, y \in D \). Show that \( f \) is constant.
**Problem 26.** Let $D \subseteq \mathbb{R}^n$ be an open set and $f : D \to \mathbb{R}^m$ be a differentiable function for which there exists a constant $M > 0$ such that

$$||f(x) - f(y)|| \leq M \cdot ||x - y||,$$

for all $x, y \in D$. Show that $||Df(x)|| \leq M$, for all $x \in D$.

**Problem 27.** Let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives and satisfy

$$\left| \frac{\partial f}{\partial x_j} \right| \leq M,$$

for all $x = (x_1, x_2, \ldots, x_n)$, $j = 1, 2, \ldots, n$. Prove that

$$|f(x) - f(y)| \leq \sqrt{n} \cdot M||x - y||.$$

**Berkeley Preliminary Exam, 1977**

**Problem 28.** Prove that the matrix function $F : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ given by $F(A) = AA^T - I$ is infinitely differentiable and show that

$$DF(A)B = BA^T + AB^T,$$

for all $B \in \mathbb{R}^{n \times n}$.

**Problem 29.** The class of invertible matrices $GL(n; \mathbb{R})$ forms an open subset in the space of all $n \times n$ matrices $\mathbb{R}^{n \times n} = \mathbb{R}^{n^2}$. The function $F : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $F(A) = A^{-1}$ is smooth and hence differentiable at any $A \in \mathbb{R}^{n \times n}$. Prove that for all $B \in \mathbb{R}^{n \times n}$,

$$DF(A)B = -A^{-1} \circ B \circ A^{-1}.$$

**Problem 30.** Let $\mathcal{M}_{n \times n}$ denote the vector space of real $n \times n$ matrices. Define the map $f : \mathcal{M}_{n \times n} \to \mathcal{M}_{n \times n}$ by $f(X) = X^2$. Find the derivative of $f$.

**Problem 31.** Define $f : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}$ by

$$f(x) = XX^T X$$

for $x \in \mathbb{R}^{2 \times 2}$. Let $I_2$ be the identity matrix. Prove that

$$Df(I)(A) = 2A + A^T,$$

for every matrix $A \in \mathbb{R}^{2 \times 2}$.

**Problem 32.** Let $\alpha \geq 0$, and

$$F_{\alpha}(x, y) = \begin{cases} \frac{x^2 y^2 + x^4 y}{(|x|^\alpha + y^2) \sqrt{x^2 + y^2}} & ,(x, y) \neq (0, 0) \\ 0 &,(x, y) = (0, 0) \end{cases}$$
Find an $\alpha_0 \in [0, \infty)$ such that

$$\lim_{(x,y) \to (0,0)} F_\alpha(x,y) = 0$$

for $0 \leq \alpha < \alpha_0$ and such that the same limit does not exist if $\alpha \geq \alpha_0$.

**Problem 33.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

If $f$ of $C^2(\mathbb{R}^2)$?

University of Missouri-Columbia Qualifying Exam, 2001

**Problem 34.** Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

has directional derivatives in all directions at $(0, 0)$ but it is not differentiable at $(0, 0)$.

University of Missouri-Columbia Qualifying Exam

**Problem 35.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} y^2 \log \left( 1 + \frac{x^2}{y^2} \right), & y \neq 0 \\ 0, & y = 0 \end{cases}$$

Show that the mixed second order partial derivatives are not continuous in the origin, but

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{\partial^2 f}{\partial x \partial y}(0, 0).$$

**Problem 36.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Show that $f$ is differentiable at $(0,0)$, but $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not continuous at $(0,0)$.

**Problem 37.** Define $f$ in $\mathbb{R}^3$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$
Show that $f(0, 1, -1) = 0$, $\frac{\partial f}{\partial x}(0, 1, -1) \neq 0$ and that there exists a differentiable function $g$ in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$ such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$ 

Find $\frac{\partial g}{\partial x}(1, -1)$ and $\frac{\partial^2 g}{\partial y^2}(1, -1)$.

**Problem 38.** Put $f(0, 0) = 0$, and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2},$$

if $(x, y) \neq (0, 0)$. Prove that

(i) $f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ are continuous in $\mathbb{R}^2$.

(ii) $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist at every point of $\mathbb{R}^2$, and are continuous except at $(0, 0)$.

(iii) $\frac{\partial^2 f}{\partial x \partial y}(0, 0) = 1$ and $\frac{\partial^2 f}{\partial y \partial x}(0, 0) = -1$.

**Problem 39.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $C^2$ and $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$F(x, y) = \begin{cases} 
  f(x) - f(y) & ,x \neq y \\
  0 & ,x = y 
\end{cases}$$

Show that $F$ is $C^1(\mathbb{R}^2)$.

**Problem 40.** Let $F$, with components $F_1, F_2, \ldots, F_n$ be a differentiable map of $\mathbb{R}^n$ into $\mathbb{R}^n$ such that $F(0) = 0$. Assume that

$$\sum_{j,k=1}^n \left| \frac{\partial F_j(0)}{\partial x_j} \right|^2 = c < 1.$$ 

Prove that there is a ball $B$ in $\mathbb{R}^n$ with center $0$ such that $F(B) \subset B$.

Berkeley Preliminary Exam, 2000

**Problem 41.** Let $A(t) = [x_{ij}(t)] : (a, b) \to \mathbb{R}^{n \times n}$ be a smooth matrix-valued curve. Prove that if $A(0) = I$, then

$$\frac{d}{dt} \bigg|_{t=0} (\det A(t)) = \sum_{i=1}^n x'_{ii}(0) = \text{tr} A'(0).$$

**Problem 42.** Let $f \in C^2(\mathbb{R}^2)$. Suppose that $\nabla f = 0$ on a compact set $E \subset \mathbb{R}^2$. Prove that there is a constant $M > 0$ such that
|f(x) - f(y)| \leq M|x - y|^2,

for all \(x, y \in E\).

**Problem 43.** Let \(F(t) = (f_{ij}(t))\) be an \(n \times n\) matrix of continuously differentiable functions \(f_{ij} : \mathbb{R} \to \mathbb{R}\), and let

\[
u(t) = \text{tr}(F(t)^3).
\]

Show that \(u\) is differentiable and

\[
u'(t) = 3 \text{tr}(F(t)F'(t)).
\]

Berkeley Preliminary Exam, 1983

**Problem 44.** Let \(M_{n \times n}\) denote the vector space of \(n \times n\) real matrices for \(n \geq 2\). Let \(\text{det} : M_{n \times n} \to \mathbb{R}\) be the determinant map.

(a) Show that \(\text{det}\) is \(C^\infty\).
(b) Show that the derivative of \(\text{det}\) at \(A \in M_{n \times n}\) is zero if and only if \(A\) has rank less or equal than \(n - 2\).

Berkeley Preliminary Exam, 1993

**Problem 45.** Let \(f : D \subseteq \mathbb{R}^2 \to \mathbb{R}\) and \((x_0, y_0) \in \text{int}(D)\). Show that if \(f\) has partial derivatives in a neighborhood of the point \((x_0, y_0)\) and if one of them is continuous at \((x_0, y_0)\), then \(f\) is differentiable in \((x_0, y_0)\).

**Problem 46.** Let the function \(f : \mathbb{R}^n \to \mathbb{R}^n\) satisfy two conditions:

(i) \(f(K)\) is compact whenever \(K\) is a compact subset of \(\mathbb{R}^n\);
(ii) If \(\{K_n\}\) is a decreasing sequence of compact subsets of \(\mathbb{R}^n\), then

\[
f \left( \bigcap_{1}^{\infty} K_n \right) = \bigcap_{1}^{\infty} f(K_n).
\]

Show that \(f\) is continuous.

Berkeley Preliminary Exam

**Problem 49.** Prove that a map \(g : \mathbb{R}^n \to \mathbb{R}^n\) is continuous only if its graph is closed in \(\mathbb{R}^n \times \mathbb{R}^n\). Is the converse true?

**Problem 50.** Let \(U \subset \mathbb{R}^n\) be an open set. Suppose that the map \(h : U \to \mathbb{R}^n\) is homeomorphism from \(U\) onto \(\mathbb{R}^n\), which is uniformly continuous. Prove that \(U = \mathbb{R}^n\).

**Problem 51.** Let \(f\) be a real valued function on \(\mathbb{R}^2\) with the following properties:

(i) for each \(y_0 \in \mathbb{R}\), the function \(x \mapsto f(x, y_0)\) is continuous.
(ii) for each \(x_0 \in \mathbb{R}\), the function \(y \mapsto f(x_0, y)\) is continuous.
(iii) $f(K)$ is compact whenever $K$ is a compact subset of $\mathbb{R}^2$.

Prove that $f$ is continuous.

**Problem 52.** A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *proper* if it is continuous and $f^{-1}(B)$ is compact for each compact subset $B$ of $\mathbb{R}^n$; $f$ is *closed* if it is continuous and $f(A)$ is closed for each closed subset of $A$ of $\mathbb{R}^m$.

(i) Prove that every proper map $f : \mathbb{R}^m \to \mathbb{R}^n$ is closed.
(ii) Prove that every one-to-one map $f : \mathbb{R}^m \to \mathbb{R}^n$ is proper.

**Problem 53.** Let $g : [0, \infty) \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = g(||x||)$ for any $x \in \mathbb{R}^n$. Show that the following statements are equivalent:

(i) $f$ is $C^1$
(ii) $g$ is $C^1$ and $g'(0) = 0$.

**Problem 54.** Let $X : (a,b) \to \mathbb{R}^{n \times n}$ be a smooth matrix-valued function. Suppose that $X(t) \in O(n)$ is an orthogonal matrix for every $t$. Suppose that there is a matrix-valued $C : (a,b) \to \mathbb{R}^{n \times n}$ such that

$$X'(t) = C(t) \cdot X(t)$$

for all $t \in (a, b)$. Prove that

$$C(t) + C^T(t) = O$$

for all $t \in (a, b)$.

**Problem 55.** Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear transformation, so $||A(x)|| = ||x||$, for every $x \in \mathbb{R}^n$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ and harmonic: $\nabla \cdot \nabla u = 0$. Prove that the composition $u \circ A$ is also harmonic.

University of Pittsburgh Preliminary Exam, 2014

**Problem 56.** For $n$ a positive integer, let $u : \mathbb{R}^n - \{0\} \to \mathbb{R}$ be a $C^2$ function. Suppose that $u$ depends only on the variable $r = \sqrt{|x|^2}$ and that $u$ is bounded on its domain. Finally suppose also that $u$ is harmonic: $\nabla \cdot \nabla u = 0$. Prove that $u$ is constant.

University of Pittsburgh Preliminary Exam, 2014