WORKSHEET FOR THE PRELIMINARY EXAMINATION-REAL ANALYSIS (SEQUENCES OF FUNCTIONS, SERIES OF FUNCTIONS & POWER SERIES)

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Problem 1. Decide which of the following sequences of functions converge uniformly and which not:

(a) \( f_n : [-1, 1] \to \mathbb{R}, f_n(x) = \frac{x}{1 + n^2 x^2}, n \geq 1. \)

(b) \( f_n : [0, 1] \to \mathbb{R}, f_n(x) = x^n(1 - x^n), n \geq 1. \)

(c) \( f_n : [0, \infty) \to \mathbb{R}, f_n(x) = \frac{x}{x + n}, n \geq 1. \)

(d) \( f_n : [a, b] \to \mathbb{R}, f_n(x) = \frac{x}{x + n}, n \geq 1. \)

(e) \( f_n : (-1, 1) \to \mathbb{R}, f_n(x) = \frac{1 - x^n}{1 - x}, n \geq 1. \)

(f) \( f_n : [-1, 1] \to \mathbb{R}, f_n(x) = e^{-x^2/n}, n \geq 1. \)

(g) \( f_n : [0, \infty) \to \mathbb{R}, f_n(x) = \frac{x}{n} e^{-x/n}, n \geq 1. \)

(h) \( f_n : [0, 1] \to \mathbb{R}, f_n(x) = nx(1 - x)^n, n \geq 1. \)

Problem 2. [Dini] Let \((f_n)_{n \geq 1}\) be a monotonic sequence of continuous functions \(f_n : [a, b] \subseteq \mathbb{R} \to \mathbb{R}\) such that \(f_n\) converges pointwise at \(f\). Show that \(f_n\) converges uniformly.

Problem 3. Let \((g_n)_n\) be a sequence of twice differentiable functions on \([0, 1]\) such that for all \(n\), \(g_n(0) = 0\) and \(g'_n(0) = 0\). Suppose that \(|g''_n(x)| \leq 1\) for all \(n, x\). Prove that there is a subsequence of \((g_n)\) which converges uniformly.

Problem 4. (a) Show that the sequence of functions \(f_n : [1, 2] \to \mathbb{R}\), defined by
\[
 f_n(x) = \frac{(\log x)^n}{1 + (\log x)^n}
\]
converges uniformly.

(b) Show that the sequence of functions \(f_n : [0, 1] \to \mathbb{R}\), defined by
\[
 f_n(x) = \frac{(1 + x)^n}{e^{2nx}}
\]
converges uniformly on any interval \([0, a]\), with \(0 < a < 1\).
Problem 5. Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ which is given by the recursion:

$$f_1 \equiv 0, f_{n+1} = f_n(x) + \frac{1}{2}[x - f_n^2(x)], x \in [0, 1], n \geq 1.$$ 

Show that $(f_n)$ converges uniformly to $f(x) = \sqrt{x}$.

Ohio State University Qualifying Exam, 2013

Problem 6. Let $f : [-a, a] \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $|f(x)| < x, x \neq 0$. Consider the sequence of functions $(f_n)_{n \geq 1}, f_n : [-a, a] \to \mathbb{R}$ defined by $f_1 = f$ and $f_{n+1} = f \circ f_n = f_n \circ f$ for all $n \geq 1$. Show that $f_n$ converges uniformly to $f = 0$ on $[-a, a]$.

Problem 7. Define the function $\zeta$ by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$ 

Prove that $\zeta(x)$ is defined and has continuous derivatives of all orders in the interval $1 < x < \infty$.

Berkeley Preliminary Exam, 1985

Problem 8. Prove that

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \log \left(1 + \frac{x}{n}\right)$$

is defined and differentiable on the open interval $-1 < x < \infty$.

Ohio State University Qualifying Exam, 2005

Problem 9. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that there exists a sequence of polynomials $P : \mathbb{R} \to \mathbb{R}$ such that $P_n$ converges uniformly to $f$.

Problem 10. For each positive integer $n$, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \cos nx$. Prove that the sequence of functions $f_n$ has no uniformly convergent subsequence.

Berkeley Preliminary Exam, 1995

Problem 11. Let the functions $f_n : [0, 1] \to [0, 1], n = 1, 2, \ldots$ satisfy

$$|f_n(x) - f_n(y)| \leq |x - y|$$

whenever $|x - y| \geq \frac{1}{n}$. Prove that the sequence $(f_n)_{n=1}^{\infty}$ has a uniformly convergent subsequence.

Berkeley Preliminary Exam, 2001
Problem 12. Prove that \( \sum_{n=1}^{\infty} \frac{x^\alpha}{\sqrt{n(n^2 + x^3)}} \) is uniformly convergent on \([0, \infty)\) if \( \alpha = 2 \), but no uniformly convergent on \([0, \infty)\) if \( \alpha = 3 \).

University of Pittsburgh Preliminary Exam, 2013

Problem 13. Determine whether the given series is uniformly convergent on \((1, \infty)\):

(a) \( \sum_{n=1}^{\infty} \frac{x^2}{n^2(n + x^2)} \).

(b) \( \sum_{n=1}^{\infty} \frac{x}{n(n + x^2)} \).

(c) \( \sum_{n=1}^{\infty} \frac{x^2}{n(n^2 + x^2)} \).

(d) \( \sum_{n=1}^{\infty} \frac{x^2}{n(n^3 + x^2)} \).

(e) \( \sum_{n=1}^{\infty} \frac{x^2}{n^2(n + x^3)} \).

Problem 14. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be differentiable for each \( n = 1, 2, \ldots \) and such that \( |f'_n(x)| \leq 1 \) for all \( n, x \). Assume \( \lim_{n \to \infty} f_n(x) = g(x) \) for all \( x \). Show that \( g \) is continuous.

University of Pittsburgh Preliminary Exam, 2009

Problem 15. Prove directly that if a sequence of continuous functions \( f_n : [0, 1] \to \mathbb{R} \) is equicontinuous and convergent at every point \( x \in [0, 1] \), then \( f_n \) is uniformly convergent on \([0, 1]\).

University of Pittsburgh Preliminary Exam, 2009

Problem 16. Prove the following:

(a) If \( a > 1 \), and \( k \geq 1 \), then
\[
\sum_{n=2}^{\infty} \frac{\log^k n}{n^a} < \infty.
\]

(b) The function \( \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \), \( x > 1 \), is infinitely differentiable in \((1, \infty)\).

University of Pittsburgh Preliminary Exam, 2009

Problem 17. Prove that the function
\[ f(x) = \sum_{n=0}^{\infty} \left( \frac{x^n}{n!} \right)^2 \]

is continuous on \( \mathbb{R} \).

University of Pittsburgh Preliminary Examination, 2005

**Problem 18.** (a) Give sufficient conditions under which the series \( \sum_{n=0}^{\infty} f_n(x) \) to be differentiated term by term on a bounded interval \( I \subset \mathbb{R} \).

(b) Can we differentiate \( \sum_{n=1}^{\infty} \arctan \frac{x}{n^2} \) term by term?

University of Pittsburgh Preliminary Exam, 2006

**Problem 19.** For \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^+ \), define

\[ f_1(x) = \sqrt{x}, f_{n+1}(x) = \sqrt{x} + f_n(x), n \geq 1. \]

Prove that \( f_n \) converges uniformly on every interval \( [a, b] \) where \( 0 < a < b < \infty \).

Ohio State University Qualifying Exam, 1999

**Problem 20.** Determine

\[ \lim_{t \to 0} \sum_{n=1}^{\infty} \frac{1}{n^2 + t^2}, \]

and

\[ \lim_{t \to \infty} \sum_{n=1}^{\infty} \frac{1}{n^2 + t^2}. \]

Ohio State University Qualifying Exam, 1995

**Problem 21.** Prove that the two series

\[ \sum_{n=0}^{\infty} c_n x^n, \]

and

\[ \sum_{n=0}^{\infty} n \log n c_n x^{n+3} \]

have the same radius of convergence.

University of Pittsburgh Preliminary Exam, 2009
Problem 22. Prove that \( \sum_{n=1}^{\infty} \frac{nx^n}{x^n + 1} \) is not uniformly convergent on \([0, 1)\), but it defines a continuous function on \([0, 1)\).

University of Pittsburgh Preliminary Exam, 2013

Problem 23. Prove that the series
\[
f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{5/2}}
\]
converges for all \( x \in \mathbb{R} \) and that \( f(x) \) is a continuous function on \( \mathbb{R} \) with continuous derivative.

UCLA Basic Examination, 2006

Problem 24. Let \( g \) be a continuous function on \( \mathbb{R} \) with \( g(0) = 0 \) and let \( g' \) be bounded on \( \mathbb{R} \), that is
\[
\sup \{|g'(x)| : x \in \mathbb{R} \} = M < \infty.
\]
(a) Show that the series
\[
\sum_{n=1}^{\infty} \frac{1}{n} g \left( \frac{x}{n} \right)
\]
converges for all \( x \in \mathbb{R} \) and that its sum
\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{n} g \left( \frac{x}{n} \right)
\]
is a continuous function on \( \mathbb{R} \).
(b) Is \( f \) differentiable on \( \mathbb{R} \)?

Problem 25. Let \( I \) be the interval \([0, \infty)\). For \( n \in \mathbb{N} \) and \( t \in I \), let
\[
f_n(t) = \sin((t + 4n^2 \pi^2)^{1/2}), n \geq 1.
\]
(i) Show that the sequence \((f_n)\) is equicontinuous on \( I \).
(ii) Show that \((f_n)\) does not contain a subsequence which is uniformly convergent on \( I \).

University of Pittsburgh Preliminary Exam, 2010

Problem 26. Let \( f_n(x) = e^{-nx} \left( 1 + \frac{x}{n} \right)^{n^2} \), defined for any real \( x \) and for any \( n \in \mathbb{N} \).
(a) Prove that there is a function $f: \mathbb{R} \to \mathbb{R}$, with $\lim_{n \to \infty} f_n(x) = f(x)$, for each real number $x$ and determine this function explicitly.

(b) Is the convergence of $f_n(x)$ to $f(x)$ uniform?

University of Pittsburgh Preliminary Exam, 2014

Problem 27. Prove the identity, valid for any real $x$, with $|x| < 1$:

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k.$$

Now let $f: (-3, 3) \to \mathbb{R}$ be given by the series, valid for any real $x$ with $|x| < 3$:

$$f(x) = \sum_{k=1}^{\infty} \left(\frac{x}{(-1)^k + 4}\right)^k.$$

Prove that $|f'(x)| \leq \frac{3}{(3-x)^2}$, for any real number $x$, such that $0 \leq x \leq 3$.

University of Pittsburgh Preliminary Exam, 2014

Problem 28. Let $f_n: [0, 1) \to \mathbb{R}$ be the function defined by

$$f_n(x) = \sum_{k=1}^{n} \frac{x^k}{1 + x^k}.$$

(a) Prove that $f_n$ converges to a function $f: [0, 1) \to \mathbb{R}$.

(b) Prove that for every $0 < a < 1$ the convergence is uniform on $[0, a]$.

(c) Prove that $f$ is differentiable on $(0, 1)$.

University of Lincoln-Nebraska Qualifying Exam, 2012

Problem 29. Let $f_n(x) = (1 - x^2)x^{2n}$, for $n = 0, 1, 2, \ldots$ and for $x \in [-1, 1]$.

(a) Show that the series $\sum_{n=0}^{\infty} f_n(x)$ converges pointwise but not uniformly, and find the limit.

(b) Show that the series $\sum_{n=0}^{\infty} (-1)^n f_n(x)$ converges uniformly on $[-1, 1]$.

University of Lincoln-Nebraska Qualifying Exam, 2006

Problem 30. (a) Consider the functions $f_n(x) = \frac{nx}{1 + n^3 x^2}$, for $x \geq 0$. Show that $f_n$ converge pointwise to zero on $[0, \infty)$. For which $a \geq 0$, if any, does $f_n$ converge uniformly on $[a, \infty)$?

(b) Suppose that a sequence of uniformly continuous functions $f_n : [0, \infty) \to \mathbb{R}$ converge uniformly to a function $f : [0, \infty) \to \mathbb{R}$. Prove that $f$ is uniformly continuous.
Problem 31. Let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ be a sequence of continuous functions on a metric space $X$ such that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in X$ and

$$\sup_{x \in X} \left( \sum_{n=1}^{\infty} f_n(x)^2 \right)^{1/2} < \infty.$$ 

Prove that if a sequence of real numbers $c_n$, $n = 1, 2, \ldots$ satisfies $\sum_{n=1}^{\infty} c_n^2 < \infty$, then the series

$$\sum_{n=1}^{\infty} c_n f_n(x)$$

converges everywhere to a continuous function.