Problem. (Putnam, 1991)

Let $A, B \in \mathcal{M}_n(\mathbb{R})$, $A \neq B$ such that $A^3 = B^3$ and $A^2B = AB^2A$. Is it possible for the matrix $A^2 + B^2$ to be invertible?

Solution.

We show that A^2+B^2 is not invertible.
Assume by contradiction that it is invertible.
$\Rightarrow \det(A^2+B^2) \neq 0$. Let's do the following:

$$(A^2+B^2)(A-B) = A^3 - A^2B + B^2A - B^3 = 0.$$

$\Rightarrow A - B = O_n \Leftrightarrow A = B$, contradiction. $\square$

Problem.

Does there exist a matrix $A \in M_n(\mathbb{Z})$ such that $\det(A^2+I_n)=2015$?

Solution.

NO! If there exists a matrix $A \in M_n(\mathbb{Z})$ such that $\det(A^2+I_n)=2015$. Using the fact that

$$A^2+I_n=(A+i\cdot I_n)(A-i\cdot I_n),$$ we have that

$$\underbrace{\det(A+i\cdot I_n)}_{\substack{\shortparallel \\ a+b\cdot i \in \mathbb{Z}[i]}}\cdot\underbrace{\det(A-i\cdot I_n)}_{\substack{\shortparallel \\ a-b\cdot i}}=2015$$

Therefore, we have $(a+b\cdot i)(a-b\cdot i)=2015$ or equivalently, $a^2+b^2=2015=5\cdot 13\cdot 31$, in contradiction with Fermat's theorem on sums of two squares, since 31 cannot be written as the sum of two squares. $\square$

Problem. (RNMO, 2014)

Consider a positive integer n and A, B two matrices in $\mathcal{M}_n(\mathbb{C})$ such that $A^2+B^2=2AB$. Prove that:

a) The matrix $AB-BA$ is <u>NOT</u> invertible;

b) If the rank of $A-B$ is 1, then matrices A and B commute.

Solution. (a) The given relation can be written in the following two forms:

$$(*)\ (A-B)^2 = AB-BA$$

$$(**)\ A(A-B) = (A-B)B.$$

Suppose $AB-BA$ is nonsingular. By $(*)$ $A-B$ is nonsingular that is $B=(A-B)^{-1}A(A-B)$ by $(**)$. Then, we have:

$$A-B = A-(A-B)^{-1}A(A-B),$$ so from here we get:

$$I_n = A(A-B)^{-1}-(A-B)^{-1}A.$$

Taking the traces, we obtain:

$$n = tr(I_n) = tr(A(A-B)^{-1}-(A-B)^{-1}A) = 0,$$

(b) So, we have that $AB-BA$ & $A-B$ are singular matrices. Since $A-B$ has rank 1, it follows that $(A-B)^2 = tr(A-B)(A-B) = AB-BA$. So, by taking the trace, we have:

$$0 = tr((A-B)^2),$$ i.e. $tr(A-B)=0$, so $AB-BA=0$ and our conclusion follows. $\square$

Problem.

Let $A \in M_n(\mathbb{C})$. Show that $tr(A^k)=0$, $(\forall) k=1,2,\dots,n$ if and only if A is nilpotent.

Solution.

"$\Rightarrow$" Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A. By the hypothesis, it follows that

$$\lambda_1^k+\lambda_2^k+\dots+\lambda_n^k=0, (\forall) k=1,2,\dots,n.$$

Recall Newton's identities

$$(-1)^m \cdot \sum_{1\le i_1<\dots<i_m\le n} x_{i_1}x_{i_2}\cdots x_{i_m} + \sum_{k=1}^{m}\left((-1)^{k+m}\left(\sum_{i=1}^{n} x_i^k\right)\cdot \sum_{1\le i_1<\dots<i_{m-k}\le n} x_{i_1}\cdots x_{i_{m-k}}\right)=0,$$

$(\forall)\, m\le n$. Using Newton's identities for $m=1,2,\dots,n$, we obtain that

$$\sum_{1\le i_1<\dots<i_m\le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_m}=0.$$

This means that the coefficients of the characteristic polynomial of A are all zero, except the dominant coefficient, so $A^n=O_n$.

"$\Leftarrow$" We show that all the eigenvalues are zero. Assume that there exist a nonzero eigenvalue $\alpha\neq 0$ and Y a nonzero corresponding eigenvector. We have

$AY=\alpha\cdot Y$ and $\alpha A^{p-1}Y=A^pY$, where p is the smallest number, so until $AY=0=\alpha\cdot Y$, contradiction. $\square$

Problem. (Linear Algebra, Putnam seminar)

Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $A^2 = B^2 = O_n$. If $A+B$ is invertible prove that

$$\operatorname{rank}(AB) + \operatorname{rank}(BA) = \operatorname{rank}(AB+BA).$$

Solution.

Firstly, let us compute

$$(A+B)(A+B) = A^2 + AB + BA + B^2 = AB + BA \quad (1)$$

As $A+B$ is invertible, the same property holds true for $(A+B)^2$. Because of this, we have $\operatorname{rank}[(A+B)^2] = n$.

Using (1), this would give us $\operatorname{rank}(AB+BA) = n$. Therefore, it is enough to prove that

$$\operatorname{rank}(AB) + \operatorname{rank}(BA) = n.$$

Now, recall Sylvester's inequality (! $\operatorname{rank}(XY) + n \geq \operatorname{rank}(X) + \operatorname{rank}(Y)$, $\forall X, Y \in \mathcal{M}_n(\mathbb{C})$!)

By Sylvester's inequality applied for $X = AB$ and $Y = BA$, we obtain:

$$\operatorname{rank}(\underbrace{ABBA}_{AB^2A = O_n}) + n \geq \operatorname{rank}(AB) + \operatorname{rank}(BA)$$

$$\Longrightarrow \operatorname{rank}(AB) + \operatorname{rank}(BA) \leq n.$$

On the other hand, using the inequality $\operatorname{rank}(X) + \operatorname{rank}(Y) \geq \operatorname{rank}(X+Y)$, for all $X, Y \in \mathcal{M}_n(\mathbb{C})$, we derive $\operatorname{rank}(AB) + \operatorname{rank}(BA) \geq \operatorname{rank}(AB+BA) = n$, so we have equality and we are done. $\square$

Remark.

In the same conditions, show that $AB-BA$ is invertible!

For this, we have:

$$(A-B)(A+B)=A^2+AB-BA+B^2=AB-BA$$

Now, we are left to prove that $A-B$ is invertible because

$$\det(AB-BA)=\det(A-B)\det(A+B)\neq 0,$$ so our problem will be solved.

On the other hand,

$$(A-B)(A-B)=A^2-AB-BA+B^2=-AB-BA.$$

Therefore,

$$\det^2(A-B)=\det(-AB-BA)=$$

$$=(-1)^n\cdot\det(AB+BA)\neq 0,$$

and thus $\det(A-B)\neq 0$ which solves our problem. ∎

Problem.

Let A, B, C be $n \times n$ real matrices that are pairwise commutative and $ABC = O_n$. Prove that

$$\det(A^3+B^3+C^3)\det(A+B+C) \geq 0.$$

Solution.

We have the identity:

$$A^3+B^3+C^3-3\underbrace{ABC}_{=O_n} = (A+B+C)(A^2+B^2+C^2-AB-BC-CA)$$

which is equivalent to

$$A^3+B^3+C^3 = (A+B+C)(A^2+B^2+C^2-AB-BC-CA).$$

This gives us the following

$$(A^3+B^3+C^3)(A+B+C) = (A+B+C)^2(A^2+B^2+C^2-AB-BC-CA)$$

Let us denote by ε the cubic root of unity. Even better, we have:

$$A^3+B^3+C^3 = (A+B+C)(A+\varepsilon B+\varepsilon^2 C)(A+\varepsilon^2 B+\varepsilon C)$$

$$= (A+B+C)(A+\varepsilon B+\varepsilon^2 C)\overline{(A+\varepsilon B+\varepsilon^2 C)}$$

$$= (A+B+C)\wedge$$

Taking determinants, this implies that

$$\det(A^3+B^3+C^3)\det(A+B+C) = \det^2(A+B+C)\cdot$$

$$\cdot \det(A+\varepsilon B+\varepsilon^2 C)\overline{(A+\varepsilon B+\varepsilon^2 C)}$$

$$= \det^2(A+B+C) + |\det(A+\varepsilon B+\varepsilon^2 C)|^2$$

$$\geq 0$$

and we are done. $\square$

Problem (Pitt Putnam Test)

Let $A_1, A_2, \ldots, A_m \in M_n(\mathbb{C})$ satisfying $A_1 + \ldots + A_m = m \cdot I_n$ and $A_1^2 = A_2^2 = \ldots = A_m^2 = I_n$. Prove that $A_1 = A_2 = \ldots = A_m$.

Solution.

Since all $A_i^2 = I_n \Rightarrow \lambda_i^2 = 1$, for all eigenvalues of $A_i \Rightarrow$ they are equal to ± 1. Since $tr(A_i) \leq n$ with equality iff all eigenvalues are equal to 1. On the other hand,

$$m \cdot n = tr(m \cdot I_n) = tr(A_1 + \ldots + A_m) =$$

$$= tr(A_1) + tr(A_2) + \ldots + tr(A_m) \leq \underbrace{n + n + \ldots + n}_{m} = m \cdot n$$

This implies that all eigenvalues of A_i are equal to 1.

Finally, since $A_i^2 = I_n \Rightarrow A_i^2 - I_n = 0$, this implies that the minimal polynomial of A_i divides $x^2 - 1$, and has no multiple roots, so that A_i is diagonalizable. But the only diagonalizable $n \times n$ matrix with all eigenvalues equal to 1 is the identity matrix, thus $A_i = I_n$, for all i. $\square$

Problem.

Let $A \in M_n(\mathbb{C})$ be a matrix of rank 1. Show that

$$A^2 = tr(A) \cdot A.$$

Solution.

It is well-known the fact that if $rank(A)=1$ then $A = xy$, where x is a column vector and y is a row vector. Then,

$$A^2 = (xy)^2 = xyxy = x(yx)y = (yx)xy$$
$$= (yx) \cdot A$$

and the conclusion follows immediately. $\square$

Problem. (RNMO, 2013)

Let A be a $n \times n$ noninvertible matrix with real entries, $n \geq 2$, and let $adj(A)$ be the adjugate matrix of A. Show that $tr(adj(A)) \neq -1$ if and only if the matrix $I_n + adj(A)$ is invertible.

Solution.

Before we start the problem, we settle the following

Lemma. Let $A \in \mathcal{M}_n(\mathbb{C})$. Then we have the following:

$$rank(adj(A)) = \begin{cases} 0, 1 & , \text{if } \det(A) = 0 \\ 1 & , \text{iff } rank(A) = n-1. \end{cases}$$

Proof of the lemma.

From Sylvester's inequality, we have

$$rank(A \cdot adj(A)) \geq rank(A) + rank(adj(A)) - n$$

On the other hand, we know that

$A \cdot adj(A) = adj(A) \cdot A = \det(A) \cdot I_n = O_n$, so

$rank(A \cdot adj(A)) = 0$, and therefore we have

$$n \geq \text{rank}(A) + \text{rank}(\text{adj}(A)).$$

<u>Case 1°.</u> If $\text{rank}(A) = n-1$, then $\text{rank}(\text{adj}(A)) \leq 1$.

Since $\text{adj}(A) \neq 0$, it follows that $\text{rank}(\text{adj}(A)) = 1$.

<u>Case 2°.</u> If $\text{rank}(A) \leq n-2$, then any minor of size $n-1$ is null, so $\text{adj}(A) = 0_n$. ∎

Going back to our problem

Clearly, $\det(A) = 0$. By the lemma, it follows that $\text{rank}(\text{adj}(A)) \in \{0, 1\}$.

If $\text{rank}(\text{adj}(A)) = 0$, it follows that $\text{adj}(A) = 0_n$, so there is nothing to prove.

If $\text{rank}(\text{adj}(A)) = 1$, by the lemma, we have $(\text{adj}(A))^2 = \text{tr}(\text{adj}(A)) \cdot \text{adj}(A)$.

"$\Longrightarrow$" Assume by contradiction that $\det(\text{adj}(A) + I_n) = 0$. This means that -1 is an eigenvalue for $\text{adj}(A)$. Since any eigenvalue is a root of the minimal polynomial of $\text{adj}(A)$ (Frobenius theorem), and thus is the root of any polynomial P such that $P(A) = 0_n$, we deduce that -1 is the root of the polynomi

$X^2 = tr(adj(A)) \cdot X$ which implies that

$(-1)^2 = tr(adj(A)) \cdot (-1) = 0$. This simply means that $tr(adj(A)) = -1$, contradiction.

"$\Leftarrow$" Assume by contradiction that $tr(adj(A)) = -1$

Since $adj(A)$ and I_n commute, we have

$$(adj(A) + I_n)^2 = (adj(A))^2 + 2\,adj(A) + I_n =$$
$$= (tr(adj(A)) + 2) \cdot adj(A) + I_n.$$

Since $tr(adj(A)) = -1$, we obtain

$$(adj(A))^2 + 2\,adj(A) + I_n = (tr(adj(A)) + 2) \cdot adj(A) + I_n = adj(A) + I_n,$$

which implies

$$(adj(A))^2 + adj(A) = O_n \text{ or}$$
$$adj(A)(adj(A) + I_n) = O_n.$$

Since $adj(A) + I_n$ is invertible, we obtain $adj(A) = O_n$, in contradiction with $tr(adj(A)) = -1$

$\square$