Problem. (Putnam, 1991)

Let $A, B \in \text{Mat}_n(\mathbb{R})$, $A \neq B$ such that $A^3 = B^3$ and $A^2 B = B^2 A$. Is it possible for the matrix $A^2 + B^2$ to be invertible?

Solution.

We show that $A^2 + B^2$ is not invertible. Assume by contradiction that it is invertible.

$\Rightarrow \det(A^2 + B^2) \neq 0$. Let's do the following:


$\Rightarrow A - B = 0 \text{ or } A = B$, contradiction. $\Box$
Problem.

Does there exist a matrix \( A \in \mathbb{M}_n(\mathbb{Z}) \) such that \( \det(A^2 + I_n) = 2015 \)?

Solution.

No! If there exists a matrix \( A \in \mathbb{M}_n(\mathbb{Z}) \) such that \( \det(A^2 + I_n) = 2015 \). Using the fact that

\[
A^2 + I_n = (A + i \cdot I_n)(A - i \cdot I_n),
\]

we have that

\[
\det(A + i \cdot I_n) \cdot \det(A - i \cdot I_n) = 2015
\]

\[
\frac{a + b \cdot i}{a - b \cdot i} \in \mathbb{C} \quad \frac{a - b \cdot i}{a + b \cdot i}
\]

Therefore, we have \((a + b \cdot i)(a - b \cdot i) = 2015\) or equivalently \(a^2 + b^2 = 2015 = 5 \cdot 31 \cdot 31\), in contradiction with Fermat's theorem on sums of two squares, since 31 cannot be written as the sum of two squares. \( \square \)
Problem. (RMMO, 2014)

Consider a positive integer $n$ and $A, B$ two matrices in $M_n(C)$ such that $A^2 + B^2 = 2AB$. Prove that:

a) The matrix $AB - BA$ is not invertible;

b) If the rank of $A - B$ is 1, then matrices $A$ and $B$ commute.

Solution. (a) The given relation can be written in the following two forms:

\[(*) \quad (A - B)^2 = AB - BA\]

\[(**): A(A - B) = (A - B)B.\]

Suppose $AB - BA$ is nonsingular. By $(*)$ $A - B$ is nonsingular, that is $B = (A - B)^{-1}A(A - B)$ by $(**)$.

Then we have:

$A - B = A - (A - B)^{-1}A(A - B)$, so from here we get:

$A^n = A(A - B)^{-1} - (A - B)^{-1}A$.

Taking the traces, we obtain:

$n = tr(0) = tr((A - B)^{-1}A - (A - B)^{-1}A) = 0$,

(b) So, we have that $AB - BA$ and $A - B$ are singular matrices. Since $A - B$ has rank 1, it follows that $\left(\frac{A-B}{2}\right)^2 = \frac{1}{2}(A+B)(A-B) = AB - BA$. So, by taking the trace, we have:

$0 = tr((A - B)^2)$, i.e. $tr(A - B) = 0$, so $AB$ and $BA$ commute.

and our conclusion follows. □
Problem:
Let \( A \in M_n(\mathbb{C}) \). Show that \( \lambda (A^k) = 0 \) for \( k = 1, 2, \ldots, n \) if and only if \( A \) is nilpotent.

Solution:
\[
\Rightarrow \quad \lambda_1, \lambda_2, \ldots, \lambda_n \text{ be the eigenvalues of } A. \text{ By the hypothesis, it follows that }
\]
\[
\lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k = 0, \quad \forall k = 1, 2, \ldots, n.
\]
Recall Newton's identities
\[
(-1)^m \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} x^{i_1} x^{i_2} \cdots x^{i_m} + \sum_{k=1}^m (-1)^k \left( \frac{m}{k} \right) \left( \sum_{x=1}^n x^k \right)^m = 0.
\]

\( \Rightarrow \quad m \in \mathbb{C} \). Using Newton's identities for \( m = 1, 2, \ldots, n \), we obtain that
\[
\sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m} = 0.
\]
This means that the coefficients of the characteristic polynomial of \( A \) are all zero, except the dominant coefficient, so \( A^n = 0 \).

\( \Leftarrow \quad \) We show that all the eigenvectors are zero.
Assume that there exist a nonzero eigenvalue \( \lambda \neq 0 \) and a corresponding eigenvector. We have \( A Y = \lambda Y \) and \( A^{k+1} Y = \lambda Y \). But if \( p \) is the smallest number so that \( A^p Y = 0 \), then \( A^p Y = 0 = \lambda Y \), contradiction. ☐
Problem. (Linear Algebra, Putnam seminar)
Let \( A, B \in \text{End}(C) \) such that \( A^2 = B^2 = 0_n \). If \( A + B \) is invertible prove that
\[
\text{rank}(AB) + \text{rank}(BA) = \text{rank}(AB + BA).
\]

Solution.

Firstly, let us compute
\[
(A + B)(A + B) = A^2 + AB + BA + B^2 = AB + BA \quad (1)
\]
As \( A + B \) is invertible, the same property holds true for \((A + B)^2\). Because of this, we have \( \text{rank}(A + B) = \text{rank}(A + B)^2 \).

Using (1), this would give us \( \text{rank}(AB + BA) = n \).

Therefore, it is enough to prove that
\[
\text{rank}(AB) + \text{rank}(BA) = n.
\]
Now, recall Sylvester's inequality (if \( \text{rank}(X) + n \geq \text{rank}(X) + \text{rank}(Y) \), or

By Sylvester's inequality applied for \( X = AB \) and \( Y = BA \), we obtain:
\[
\text{rank}(AB + BA) + n \geq \text{rank}(AB) + \text{rank}(BA)
\]
\[
AB + BA = 0_n \Rightarrow \text{rank}(AB) + \text{rank}(BA) \leq n.
\]
On the other hand, using the inequality \( \text{rank}(X) + \text{rank}(Y) \geq \text{rank}(X + Y) \), for all \( X, Y \in \text{End}(C) \), we have
\[
\text{rank}(AB) + \text{rank}(BA) \geq \text{rank}(AB + BA) = n,
\]
so we have equality and we are done. \( \square \)
Remark.

In the same conditions, show that $AB-BA$ is invertible.

For this, we have:

$$(A-B)(A+B) = A^2 + AB - BA + B^2 = AB - BA$$

Now, we are left to prove that $A-B$ is invertible because

$$\det(AB-BA) = \det(A-B) \det(A+B) \neq 0,$$

so our problem will be solved.

On the other hand,


Therefore,

$$\det^2(A-B) = \det(-AB-BA) = (-1)^n \det(AB+BA) \neq 0$$

and thus $\det(A-B) \neq 0$ which solves our problem.
Problem.

Let $A, B, C$ be $n \times n$ real matrices that are pairwise commutative and $ABC = O_n$. Prove that

$$\det (A^3 + B^3 + C^3) \det (A + B + C) \geq 0.$$

Selection.

We have the identity:

$$A^3 + B^3 + C^3 - 3ABC = (A + B + C)(A^2 + B^2 + C^2 - AB - BC - CA),$$

which is equivalent to


This gives us the following


Let us denote by $\varepsilon$ the cubic root of unity. Even better, we have:

$$A^3 + B^3 + C^3 = (A + B + C)(A + \varepsilon B + \varepsilon^2 C)(A + \varepsilon^2 B + \varepsilon C) = (A + B + C)(A + \varepsilon B + \varepsilon^2 C)(A + \varepsilon^2 B + \varepsilon C)$$

Taking determinants, this implies that
\[ \det (A^3 + B^3 + C^3) \cdot \det (A + B + C) = \det^2 (A + B + C). \]
\[ \cdot \det (A + \varepsilon B + \varepsilon^2 C)(A + \varepsilon B + \varepsilon C) \]
\[ = \det^2 (A + B + C) + \det (A + \varepsilon B + \varepsilon^2 C)^2 \]
\[ \geq 0 \]
and we are done. \( \square \)
Problem (Pitt Putnam Test)

Let $A_1, A_2, \ldots, A_m \in \mathbb{M}_n(\mathbb{C})$ satisfying $A_1 + \cdots + A_m = m \cdot I_n$ and $A_1^2 = A_2^2 = \cdots = A_m^2 = I_n$. Prove that $A_1 = A_2 = \cdots = A_m$.

Solution:

Since all $A_i^2 = I_n \Rightarrow x_i^2 = 1$, for all eigenvalues of $A_i$ they are equal to $\pm 1$. Since $\text{tr}(A_i) \leq n$ with equality iff all eigenvalues are equal to $1$. On the other hand,

$$m \cdot n = \text{tr}(m \cdot I_n) = \text{tr}(A_1 + \cdots + A_m) = \text{tr}(A_1) + \text{tr}(A_2) + \cdots + \text{tr}(A_m) \leq n + n + \cdots + n = m \cdot n$$

This implies that all eigenvalues of $A_i$ are equal to $1$.

Finally, since $A_i^2 = I_n \Rightarrow A_i - I_n = 0$, this implies that the minimal polynomial of $A_i$ divides $x^2 - 1$ and has no multiple roots, so that $A_i$ is diagonalizable. But the only diagonalizable $n \times n$ matrix with all eigenvalues equal to $1$ is the identity matrix, thus $A_i = I_n$ for all $i$. $\square$
Problem. Let $A \in \mathbb{M}_n(\mathbb{C})$ be a matrix of rank 1. Show that $A^2 = \text{tr}(A) \cdot A$.

Solution. It is well-known the fact that if rank $A = 1$, then $A = xy^*$, where $x$ is a column vector and $y$ is a row vector. Then,

$$A^2 = (xy)^2 = xy \cdot xy = y^* x (yx) y = (yx) x y = (y x)^* A$$

and the conclusion follows immediately. $\square$
Problem. (RMMO, 2013)

Let $A$ be a $n \times n$ non-invertible matrix with real entries, $n \geq 2$, and let $\text{adj}(A)$ be the adjugate matrix of $A$. Show that $\text{tr}(\text{adj}(A)) + 1$ if and only if the matrix $I_n + \text{adj}(A)$ is invertible.

Solution.

Before we start the problem, we settle the following lemma. Let $A \in M_n(\mathbb{C})$. Then we have the following:

$$\text{rank}(\text{adj}(A)) = \begin{cases} 0, & \text{if } \det(A) = 0 \\ 1, & \text{iff } \text{rank}(A) = n-1. \end{cases}$$

Proof of the lemma.

From Sylvester's inequality, we have

$$\text{rank}(A \cdot \text{adj}(A)) \leq \text{rank}(A) + \text{rank}(\text{adj}(A)) - n.$$ 

On the other hand, we know that $A \cdot \text{adj}(A) = \det(A) \cdot I_n = \det(A) \cdot I_n = 0$, so

$$\text{rank}(A \cdot \text{adj}(A)) = 0,$$ 

and therefore we have

$$\text{rank}(A \cdot \text{adj}(A)) = 0.$$
\[ n \geq \text{rank}(A) + \text{rank}(\text{adj}(A)) \]

**Case 1°.** If \( \text{rank}(A) = n-1 \), then \( \text{rank}(\text{adj}(A)) \leq 1 \).

Since \( \text{adj}(A) \neq 0 \), it follows that \( \text{rank}(\text{adj}(A)) = 1 \).

**Case 2°.** If \( \text{rank}(A) \leq n-2 \), then any minor of size \( n-1 \times n-1 \) is null, so \( \text{adj}(A) = 0_n \).

Going back to our problem.

Clearly, \( \det(A) = 0 \). By the lemma, it follows that \( \text{rank}(\text{adj}(A)) \notin \{0, 1\} \).

If \( \text{rank}(\text{adj}(A)) = 0 \), it follows that \( \text{adj}(A) = 0 \).

So there is nothing to prove.

If \( \text{rank}(\text{adj}(A)) = 1 \), by the lemma, we have \( (\text{adj}(A))^2 = 0_2 (\text{adj}(A)) \cdot \text{adj}(A) \).

\[ \Rightarrow \text{ Assume by contradiction that } \det(\text{adj}(A) + I_n) = 0. \] This means that \(-1\) is an eigenvalue for \( \text{adj}(A) \). Since \(-1\) is a root of the minimal polynomial of \( \text{adj}(A) \) (Frobenius theorem), and thus is the root of any polynomial \( p \) such that \( p(A) = 0_n \), we deduce that \(-1\) is the root of the polynomial.
$X^2 = \text{tr} (\text{adj}(A)) \cdot X$ which implies that

$(-1)^2 = \text{tr} (\text{adj}(A)) \cdot (-1) = 0$. This simply means that $\text{tr} (\text{adj}(A)) = -1$, contradiction.

"Assume by contradiction that $\text{tr}(\text{adj}(A)) = -1$.

Since $\text{adj}(A)$ and $I_n$ commute, we have

$(\text{adj}(A) + I_n)^2 = (\text{adj}(A))^2 + 2 \cdot \text{adj}(A) + I_n = (-1 + 2) \cdot \text{adj}(A) + I_n.$

Since $\text{tr}(\text{adj}(A)) = -1$, we obtain

$(\text{adj}(A))^2 + 2 \cdot \text{adj}(A) + I_n = (-1 + 2) \cdot \text{adj}(A) + I_n = \text{adj}(A) + I_n,$

which implies

$(\text{adj}(A))^2 + \text{adj}(A) = 0_n \text{ or } \text{adj}(A)(\text{adj}(A) + I_n) = 0_n.$

Since $\text{adj}(A) + I_n$ is invertible, we obtain $\text{adj}(A) = 0_n$, in contradiction with $\text{tr}(\text{adj}(A)) = -1$.

$\square$