Continuity & Differentiability

$C([a,b]) = \text{space of all continuous functions}$
$C^1([a,b]) = \text{space of all differentiable functions with continuous derivative}$

Theorem (Intermediate Value Property)
Any $f \in C([a,b])$ has IVP \( f(a) \neq f(b) \)

Theorem
$f \in C([a,b]) / f$ injective $\implies f$ strictly monotone

Theorem
$f$ monotone + IVP $\implies f$ continuous

Example
Show that there are no function $f \in C(\mathbb{R})$ such that $f(f(x)) = -x$, $\forall x \in \mathbb{R}$

Solution
Assume by contradiction that $\exists f \in C(\mathbb{R})$ with $f(f(x)) = -x$, $\forall x \in \mathbb{R}$

Question: Is $f$ injective? Yes!

Exercise

$f(x_1) = f(x_2) \implies x_1 = x_2$ in contradiction with the fact $-x$ is decreasing.

Arg that the function $(f \circ f)(x) = f(f(x))$ is increasing.
Problem (CRMO, 1983) & Putnam 1991

Let \(a, b \in \mathbb{R}, \quad a, b \in (0, \frac{1}{2}) \) and \(f \in \mathbb{C}^{CR} \) such that

\[ f(f(x)) = af(x) + bx, \quad \forall x \in \mathbb{R} \]

\[ f(0) = 0 \implies f \text{ has a fixed point!} \]

and moreover, \(0\) is that fixed point.

First, let's prove that \(f\) has a fixed point!

\[ \exists x_0 \in \mathbb{R} \text{ such that } f(x_0) = x_0 \implies a + b < 1 \]

\[ x_0 = 0 ? \]

**Question:** Is \(f\) injective?

\[ f(x_1) = f(x_2) \implies x_1 = x_2 \]

\[ x_0 = 0 \]

**Assume by contradiction that** \(f(x) \neq x\), \(\forall x \in \mathbb{R}\)

**Case 1:** \(f\) increasing: \(f(f(x)) > f(x)\)

\(a f(x) + bn > f(x)\) or \(af(x) < bx\)

\((1 - a)f(x) < b(x) \rightarrow \text{false}\)

**Case 2:** \(f\) decreasing: \(f(f(x)) < f(x)\)

\((1 - a)f(x) > b(x), \forall x \rightarrow af(x) + bx < f(x)\)

\(\text{nothing wrong with it, more needs to be done}\)

\[ f(x) > \frac{b}{1 - a} \quad \text{as } x \to \infty \]
Example (HW!) \( f \in \mathbb{C} [\mathbb{R}] \) such that

\[ f(x) = f(x^2), \quad \forall x \in \mathbb{R}. \Rightarrow f \text{ is constant} \]

\[ f(x) \text{ Hint: } f(x) = f(x^{2^n}) \]

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Differentiability — Mean Value Theorem (Lagrange)

\( f: [a, b] \to \mathbb{R} \) differentiable on \([a, b]\)

and continuous on \((a, b)\), then \( \exists c \in (a, b) \)

such that

\[ \frac{f(b) - f(a)}{b - a} = f'(c) \]

- Rolle's theorem — if in addition \( f(a) = f(b) \)

\[ \Rightarrow c \in (a, b) \text{ such that } f'(c) = 0. \]

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Problem (Putnam, 2015) such that \( f \) has at least 5 distinct zeroes

\[ f \text{ 3-times differentiable on } \mathbb{R} \text{ and real-valued} \]

\[ f + 6f' + 12f'' + 8f''' \text{ has at least 2 distinct zeroes.} \]

Problem (related) \( f \) differentiable such that \( f(0) = 0 \)

\[ \Rightarrow \exists x_0 \in \mathbb{R} \text{ such that } f(x_0) = f(1) \]

\[ f(x) = c, \quad e^x \quad \text{for some } c \in (0, 1) \]

Solution

\( H: \mathbb{R} \to \mathbb{R}, \quad H(x) = e^x, \quad f(x) \text{ differentiable} \)

\[ H(0) = f(0) = 0 \quad \text{Rolle's Theorem such that } H'(c) = 0 \]

\[ H'(x) = (e^x f(x))' = e^x (f'(x) - f(x)) \]

\[ f'(c) = f(c) \text{ for some } c \in (0, 1) \]
Consider the function, $g(x) = C \cdot e^{ax} \cdot f(x), \ x \in \mathbb{R}$

$g \in C^3(\mathbb{R}) \implies g'''(x) = ?$

If the zeros of $g$, $g'(x) = C e^{ax} (2f(x) + f'(x))$

$g''(x) = C e^{ax} (x^2f(x) + 2xf'(x) + f''(x))$

Use Rolle's Theorem

$g'''(x) = C e^{ax} (x^3f(x) + 3x^2f'(x) + 3xf''(x) + f''')$

$\lambda := \frac{1}{2} \frac{C e^{ax}}{8} (f(x) + 6f'(x) + 12f''(x) + 8f''')$

Problem (Putnam B5, 2011)

Is there a strictly increasing $f : \mathbb{R} \to \mathbb{R}$ such that $f''(x) = f(f(x))$, for all real $x$?

Not enough information from the first derivative $\implies$ use the second derivative

$f'''$ is non-decreasing

Case 1 $f'''(x) = 0$ $\implies$ $f(x) = ax + b$ (V) $\forall x \in \mathbb{R}$

$\implies f(f(x)) = a$ $\forall x$

Case 2 $f'''(x) > 0$ $\implies f''(x)$ is increasing

(Homework) (Homework)