1. Riemann Sums.

The definite integral of a function is the area under the graph of the function.

Riemann's idea was to approximate the area under the graph by a family of rectangles. The sum of the areas of the rectangles will be called the Riemann sum. When these rectangles have equal width, the approximation of the Riemann sum reads as the following

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k) = \int_{a}^{b} f(x)dx,$$

where each number $\xi_k$ belongs to the interval $[a + \frac{k}{n}(b-a), a + \frac{k+1}{n}(b-a)]$.

In the following problem, we shall see how useful is to think of a Riemann sum.
Example.
Compute the following limit
\[ \lim_{n \to \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} \right) \]

Solution.
Indeed, the main idea is to rewrite the sequence
\[ x_n = \sum_{k=1}^{n} \frac{1}{n+k} \] as a Riemann sum!

We have
\[ x_n = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{k}{n}}. \]

Consider the function \( f : [0,1] \to \mathbb{R}, \quad f(x) = \frac{1}{1+x} \)

associated to the division
\[ \Delta : \quad x_0 = \frac{0}{n} < x_1 = \frac{1}{n} \leq \ldots \quad x_{k-1} = \frac{k-1}{n} < x_k = \frac{k}{n} < \ldots < x_n = \frac{n}{n}. \]

The norm of the division \( \Delta \) is given by
\[ \| \Delta \| = \max_{0 \leq k \leq n} (x_k - x_{k-1}) = \frac{1}{n} \]

\[ \lim_{n \to \infty} \| \Delta \| = 0 \]

Consider the intermediate points \( x_k = \frac{k}{n} \in [x_k, x_{k+1}] \) and we form the Riemann sum
\[ \frac{1}{n} \sum_{k=1}^{n} f\left( \frac{k}{n} \right) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{k}{n}} \]

\[ \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{1 + \frac{k}{n}} \to \int_{0}^{1} \frac{1}{1+x} \, dx = \ln(1+x)|_{0}^{1} = \ln 2. \]
Theorem.
Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function and $F: [a, b] \rightarrow \mathbb{R}$ given by
\[ F(x) = \int_a^x f(t) \, dt, \]
for all $x \in [a, b]$. Then $F$ is continuous.

Theorem.
Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable and continuous function at the point $x_0$.

The function
\[ F(x) = \int_a^x f(t) \, dt, \]
for all $x \in [a, b]$, is differentiable at $x_0$, then
\[ F'(x_0) = f(x_0). \]

Consequence.
Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function on the interval $[a, b]$.

The function $F: [a, b] \rightarrow \mathbb{R}$,
\[ F(x) = \int_a^x f(t) \, dt \]
is differentiable for all $x \in [a, b]$ and $F' = f$. 
First mean value theorem.
Let \( f: [a,b] \rightarrow \mathbb{R} \) be a continuous function. Then there is some \( c \in (a,b) \) such that
\[
\int_{a}^{b} f(x) \, dx = (b-a) \cdot f(c).
\]

Second mean value theorem.
Let \( f: [a,b] \rightarrow \mathbb{R} \) be a continuous function and \( g: [a,b] \rightarrow \mathbb{R} \) is a positive and integrable function. Then there is some \( c \in (a,b) \) such that
\[
\int_{a}^{b} f(x) g(x) \, dx = f(c) \cdot \int_{a}^{b} g(x) \, dx.
\]

Integral inequalities.
1. Cauchy-Schwarz's inequality.
\[
\left( \int_{a}^{b} f(x) g(x) \, dx \right)^2 \leq \int_{a}^{b} f^2(x) \, dx \cdot \int_{a}^{b} g^2(x) \, dx,
\]
for \( f, g: [a,b] \rightarrow \mathbb{R} \) integrable. Equality holds iff \( f = \lambda \cdot g \) for \( \lambda \in \mathbb{R} \).

2. Minkowski's inequality
\[
\left( \int_{a}^{b} \sqrt{f(x) + g(x)} \, dx \right)^{\frac{1}{2}} \leq \left( \int_{a}^{b} f(x)^{1/2} \, dx \right)^{\frac{1}{2}} + \left( \int_{a}^{b} g(x)^{1/2} \, dx \right)^{\frac{1}{2}},
\]
for \( f, g: [a,b] \rightarrow \mathbb{R} \) integrable functions.
3. Hölder's inequality

\[ \left| \int_a^b f(x)g(x)\,dx \right| \leq \left( \int_a^b |f(x)|^p \,dx \right)^{1/p} \left( \int_a^b |g(x)|^q \,dx \right)^{1/q} \]

for \( f, g : [a,b] \to \mathbb{R} \) integrable functions.

4. Chebyshev's inequality

(i) \( \frac{1}{b-a} \int_a^b f(x)g(x)\,dx \geq \left( \frac{1}{b-a} \int_a^b f(x)\,dx \right) \left( \frac{1}{b-a} \int_a^b g(x)\,dx \right) \)

if \( f, g : [a,b] \to \mathbb{R} \) are increasing.

(ii) \( \frac{1}{b-a} \int_a^b f(x)g(x)\,dx \leq \left( \frac{1}{b-a} \int_a^b f(x)\,dx \right) \left( \frac{1}{b-a} \int_a^b g(x)\,dx \right) \)

if \( f, g : [a,b] \to \mathbb{R} \) are not increasing or decreasing simultaneously.

**Definition.**

Let \((f_n)_{n=1}^{\infty} : [a,b] \to \mathbb{R}\) be a sequence. We say that \(f_n\) **converges uniformly** to the function \(f : [a,b] \to \mathbb{R}\) if for every \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that for all \(n \geq N\) and any \(x \in [a,b]\),

\[ |f_n(x) - f(x)| < \varepsilon. \]

A very useful criterion for uniform convergence is...
the following

**Proposition.**

Let \((f_n)_{n=1}^\infty : [a,b] \to \mathbb{R}\) be a sequence of functions. Then the sequence \((f_n)\) converges uniformly to \(f : [a,b] \to \mathbb{R}\) iff

\[
\lim_{n \to \infty} \sup_{x \in [a,b]} |f_n(x) - f(x)| = 0.
\]

**Theorem.**

Let \((f_n)_{n=1}^\infty : [a,b] \to \mathbb{R}\) be a sequence of continuous functions and \(f : [a,b] \to \mathbb{R}\) is a function. If \(f_n \to f\), then we have the following (\(f_n\) converges uniformly to \(f\))

\[
\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx = \int_a^b \lim_{n \to \infty} f_n(x) \, dx.
\]

**Problems.**

1. Let \((a_n)_{n=1}^\infty\) be a sequence defined by

\[
a_n = \int_0^\infty \log(1+e^{-x}) \, dx.
\]

Show that the sequence converges and show that its limit is in the interval \([\frac{3}{4}, 1\]).
2. Let $f \in C^4([0,1])$ be real-valued such that $f(0) = f(1) = 0$. Show that
\[ \int_0^1 (f(x))^2 \, dx \geq 12 \left( \int_0^1 f' \, dx \right)^2. \]

3. Let $f \in C([0,1])$ be real-valued such that $\int_0^1 f(x) \, dx = \int_0^1 x f(x) \, dx = 1$. Show that $\int_0^1 f(x)^2 \, dx \geq 4$.

4. Let $f \in C^1([0,1])$ such that $\int_0^1 f(x) \, dx = \int_0^1 x f(x) \, dx = 1$. Show that $\int_0^1 (f'(x))^2 \, dx \geq 30$.

5. Let $f : [0,1] \to \mathbb{R}$ be differentiable such that $\int_0^1 f(x) \, dx = \int_0^1 x f(x) \, dx = 0$. Show that $f'$ has at least one zero in $(0,1)$.

6. Let $f \in C([0,1])$ such that $\int_0^1 f(x) \, dx = 0$. Show that there is some $c \in (0,1)$ such that $\int_0^c x f(x) \, dx = 0$. 


7. Let \( f : [0,1] \to \mathbb{R} \) be continuous and differentiable on \((0,1)\). If there exists \( a \in (0,1) \) such that \( \int_0^a f(x) \, dx = 0 \), then show that
\[
\int_0^1 |f(x)| \, dx \leq \frac{1}{2} \cdot \sup_{x \in (0,1)} |f(x)|.
\]

8. Let \( f \in C^1([0,1]) \) such that \( \int_0^1 f(x) \, dx = 0 \). Show that for all \( \varepsilon \in (0,1) \)
\[
\int_0^1 |f(x)| \, dx \leq \frac{\varepsilon}{\varepsilon} \cdot \max_{x \in [0,1]} |f(x)|.
\]

9. Let \( f : [0, \infty) \to \mathbb{R} \) and \( g : [0,1] \to \mathbb{R} \) be continuous functions such that \( \lim_{x \to \infty} f(x) = L \in \mathbb{R} \). Show that
\[
\lim_{n \to \infty} \int_0^1 \frac{1}{n} \sum_{k=0}^{n-1} f(x + \frac{k}{n}) \, dx = L \cdot \int_0^1 g(x) \, dx.
\]