An introduction about liminf and limsup of sequences of real numbers.

Monotone sequences in \( \mathbb{R} \) always have a proper or improper limit. A proper limit means that the limit exists in \( \mathbb{R} \), and an improper limit means the sequence diverges to \( +\infty \) or \( -\infty \).

Using that fact we define the inferior and superior limits of an arbitrary sequence \( (x_n)_{n \geq 1} \) in \( \mathbb{R} \) as follows. For \( n \in \mathbb{N} \), we set

\[
\begin{align*}
\alpha_n &:= \inf \{ x_n, x_{n+1}, x_{n+2}, x_{n+3} \} = \inf_{k \geq n} x_k, \\
\beta_n &:= \sup \{ x_n, x_{n+1}, x_{n+2}, x_{n+3} \} = \sup_{k \geq n} x_k.
\end{align*}
\]

Since \( \{ x_{n+1}, x_{n+2}, x_{n+3} \} \subseteq \{ x_n, x_{n+1}, x_{n+2}, x_{n+3} \} \) and properties of infimum and supremum imply that

\( (\alpha_n) \) is increasing and \( (\beta_n) \) is decreasing. Hence,

\[
\liminf x_n = \lim_{n \to \infty} \alpha_n = \lim_{k \to \infty} \inf x_k
\]

\[
\limsup x_n = \lim_{n \to \infty} \beta_n = \lim_{k \to \infty} \sup x_k.
\]

Exist either as a proper or as an improper limit.

Below, we list some facts about inferior and superior limits:
\[
\liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n;
\]

\[
\liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n \text{ if and only if } \lim_{n \to \infty} x_n \text{ exists (as a proper or improper limit).}
\]

If that is the case all three are equal. We call a sequence \((x_{n_k})_{k \in \mathbb{N}}\) or simply \((x_{n_k})\) a subsequence of \(x_n\) if \((n_k)\) is strictly increasing and \(n_k \to \infty\). Subsequences have the following properties:

- The limit inferior of \((x_n)\) coincides with the smallest accumulation point of \((x_{n_k})_{k \in \mathbb{N}}\) or \(-\infty\).
- The superior limit of \((x_n)\) coincides with the largest accumulation point of \((x_{n_k})_{k \in \mathbb{N}}\) or \(+\infty\).
- Every bounded sequence has a convergent subsequence.

**Ex. 1.** Consider the sequence

\[
x_n = \frac{3 \cdot (-1)^n \cdot n^2}{n^2 - n + 1}, \quad n \geq 0.
\]

Find \(\liminf_{n \to \infty} x_n\) and \(\limsup_{n \to \infty} x_n\).

**Solution.** Let's look at the sequence \(x_n\). We have

\[
|x_n| = \frac{3n^2}{n^2 - n + 1} = \frac{3}{\left(\frac{4}{n - \frac{1}{2}}\right)^2 + \frac{3}{4}} - 2^-,
\]

for all \(n \geq 1\).
Since \((\frac{4}{n} - \frac{1}{2})^2\) is increasing for \(n \geq 2\) it follows that \(|x_n|\) is decreasing for \(n \geq 2\). Moreover, \(x_0 = 0\), \(x_1 = -3\), \(x_2 = 4\) and \(x_3 = -\frac{27}{7}\). Now, we need to find

\[a_n = \inf_{k \geq n} x_k\text{ and } b_n = \sup_{k \geq n} x_k.\]

Clearly,

\[a_n = \begin{cases} \frac{-27}{7}, & \text{for } n = 0, 1 \\ x_{n+1}, & \text{for } n \geq 2 \text{ even} \\ x_n, & \text{for } n \geq 3 \text{ odd} \end{cases}\]

\[b_n = \begin{cases} 4, & \text{for } n = 0, 1 \\ x_n, & \text{for } n \geq 2 \text{ even} \\ x_{n+1}, & \text{for } n \geq 3 \text{ odd} \end{cases}\]

Since \(\lim_{n \to \infty} \frac{3}{1 - \frac{1}{n} + \frac{1}{n^2}} = 3\), it follows that

\[\liminf_{n \to \infty} x_n = \lim_{n \to \infty} x_{2n+1} = -3\text{ and } \limsup_{n \to \infty} x_n = \lim_{n \to \infty} x_{2n} = 3.\]

Ex. 2.

Compute the inferior limit and the superior limit of the sequence

\[x_n = \begin{cases} \frac{n}{4}, & n \text{ even} \\ -\frac{3}{n}, & n \text{ odd} \end{cases}\]
We clearly have

\[ a_n = \inf_{k \geq n} x_k = \inf_{k \geq n} \frac{1}{k} = 0, \text{ for all } n \geq 0. \]

Hence,

\[ \liminf_{n \to \infty} x_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} 0 = 0. \]

Since the sequence is not bounded from above, \( \limsup_{n \to \infty} x_n = \infty. \) We illustrate identify the possible limits of convergent subsequences. From the definition of the sequence we see that \( \lim_{n \to \infty} x_{2n} = \infty \) and \( \lim_{n \to \infty} x_{2n+1} = 0. \) Hence, any subsequence of \( (x_n)_n \) which has a limit will either tend to \( \infty \) or to \( 0 \) as \( n \to \infty. \) Hence, \( \liminf_{n \to \infty} x_n = 0 \) and \( \limsup_{n \to \infty} x_n = \infty. \)

That is the same as we obtained before.

Ex 3. Compute the inferior and superior limits of the sequence

\[ x_n = \begin{cases} 
(-4)^{n/2} \cdot \frac{a}{n+1}, & n \text{ even} \\
\frac{n^2 - 1}{2n^2 + 1}, & n \text{ odd}
\end{cases} \]

Solution.

We identify the accumulation point of \( (x_n)_n \) by looking for convergent subsequences. We note that
\[ X_{4n} \xrightarrow{n \to \infty} 1 \quad , \quad X_{4n+1} \xrightarrow{n \to \infty} \frac{1}{2} \]

\[ X_{4n+2} \xrightarrow{n \to \infty} -1 \quad , \quad X_{4n+3} \xrightarrow{n \to \infty} \frac{1}{2} \]

Any convergent subsequence has either limit 1, -1 or \( \frac{1}{2} \). As the smallest and the largest are -1 and 1, respectively, we have

\[ \liminf_{n \to \infty} X_n = -1 \quad , \quad \limsup_{n \to \infty} X_n = 1. \]

\textbf{Problem.}

Suppose that \( (a_n)_n \) is a sequence in \( \mathbb{R} \) with \( a_n \to 0 \) for all \( n \in \mathbb{N} \). If \( (\frac{a_{n+1}}{a_n})_n \) is a bounded sequence, prove that

\[ \liminf_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf_{n \to \infty} \sqrt[n]{a_n} \leq \limsup_{n \to \infty} \sqrt[n]{a_n} \leq \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|. \]

\textbf{Solution.}

We only prove the last inequality. The middle one is obvious from the definition of the inf and sup, reversing the signs, and the first one can be obtained by
Since \( \left( \frac{a_{n+1}}{a_n} \right)_{n \in \mathbb{N}} \) is bounded, we have that
\[
S := \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]
exists in \( \mathbb{R} \). Now, let us fix \( \varepsilon > 0 \). Then by definition of the superior limit there exists \( N \geq 1 \) (depending on \( \varepsilon \)) such that
\[
\sup_{k \geq n} \left| \frac{a_{k+1}}{a_k} \right| < \varepsilon + \varepsilon, \text{ for all } n \geq N.
\]
In particular, \( \left| \frac{a_{k+1}}{a_k} \right| < \varepsilon + \varepsilon \) for all \( k \geq N \). Given \( n > N \) we apply the inequality for each \( k = N, \ldots, n-1 \) and we have
\[
\left| \frac{a_n}{a_N} \right| = \left| \frac{a_n}{a_1} \cdot \frac{a_1}{a_2} \cdot \cdots \cdot \frac{a_{N-1}}{a_N} \right| \leq (\varepsilon + \varepsilon)^{n-N}.
\]
Therefore, we get
\[
\sqrt[n]{\left| \frac{a_n}{a_N} \right|} \leq (\varepsilon + \varepsilon)^{\frac{n-N}{n}} \sqrt[n]{\varepsilon + \varepsilon}, \text{ for all } n > N.
\]
Also, we know that \( \sqrt[n]{\varepsilon + \varepsilon} \rightarrow 1 \) as \( n \to \infty \) and similarly \( (\varepsilon + \varepsilon)^{\frac{N}{n}} \rightarrow 1 \) as \( n \to \infty \). Therefore, we have
\[
\limsup_{n \to \infty} \sqrt[n]{\left| \frac{a_n}{a_N} \right|} \leq \varepsilon + \varepsilon.
\]
The above argument works for every choice of $\varepsilon > 0$, and hence $\limsup_{n \to \infty} \sqrt[n]{a_n} \leq 0$ as claimed. \( \square \)

**Problem.**

Let $(x_n)_{n \geq 1}$ be a decreasing sequence of positive terms such that the series $\sum_{n \geq 1} x_n$ is convergent. Show that $\lim_{n \to \infty} nx_n = 0$.

**Solution 1.**

We show that

\[
\begin{align*}
\begin{cases}
2n x_{2n} & \xrightarrow{n \to \infty} 0 \\
(2n+1) x_{2n+1} & \xrightarrow{n \to \infty} 0
\end{cases}
\Rightarrow nx_n \xrightarrow{n \to \infty} 0
\end{align*}
\]

i) Since the series $\sum_{n \geq 1} x_n$ is convergent it follows that $\lim_{n \to \infty} \sum_{k \geq n} x_k = 0$. In our case, we have $x_n$ is decreasing.

\[
2nx_{2n} = 2 \cdot nx_{2n} = 2 \left( x_{2n} + x_{2n+1} + \ldots + x_{2n} \right) \leq \frac{2}{n} (x_{2n+1} + x_{2n+2} + \ldots + x_{2n}).
\]

Therefore we obtained the following inequality:
0 < 2n \cdot x_{2n} \leq 2 \cdot \sum_{k=n+1}^{2n} x_k

By the squeeze theorem, it follows that
\[ 2n \cdot x_{2n} \xrightarrow{n \to \infty} 0. \]

ii) We have
\[ 0 < (2n+1) \cdot x_{2n+1} = 2n \cdot x_{2n+1} + x_{2n+1} \leq 2n \cdot x_{2n} + x_{2n+1} \]

By the squeeze theorem again, it follows that
\[ \lim_{n \to \infty} (2n+1) \cdot x_{2n+1} = 0 \] and the conclusion follows.

Problem.
Let \((a_n)_{n=1}^{\infty}\) be a sequence of real numbers such
that the series $\sum_{n=1}^{\infty} a_n^2$ is convergent.

Show that

$$\lim_{{n \to \infty}} \frac{a_1 + a_2 + \ldots + a_n}{n} = 0.$$  

Solution.

Since $\sum_{n=1}^{\infty} a_n^2$ converges, it follows that the sequence $\sum_{k=1}^{n} a_k^2$ is also convergent. On the other hand, by the Cauchy-Schwarz's inequality, we have

$$0 \leq \frac{a_1 + a_2 + \ldots + a_n}{n} \leq \sqrt{\frac{\left(\sum_{k=1}^{n} a_k^2\right)^2}{n^2}} \leq \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \to 0 \quad \text{as} \quad n \to \infty.$$  

By the squeeze theorem, we are done. \(\square\)
Problem.

Study the convergence of the following series:

a) $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$;  

b) $\sum_{n=1}^{\infty} \left(1 - \cos \frac{\pi}{n}\right)$.

Solution.

In both cases we use the limit comparison test.

a) We know that

$$\sin u \sim u, \text{ as } u \to 0$$

$$\left(\lim_{u \to 0} \frac{\sin u}{u} = 1\right)$$

so by taking $u = \frac{1}{n}$, we have that

$$\frac{1}{n} \sin \frac{1}{n} \sim \frac{1}{n^2},$$

and thus

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n} \sim \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ convergent},$$

so $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$ is convergent.

b) We know that

$$1 - \cos x \sim \frac{x^2}{2}, \text{ as } x \to 0,$$

so by taking $x = \frac{\pi}{n}$, we get

$$1 - \cos \frac{\pi}{n} \sim \frac{\pi^2}{2n^2},$$

and thus

$$\sum_{n=1}^{\infty} \left(1 - \cos \frac{\pi}{n}\right) \sim \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ convergent}.$$
Abel-Dirichlet criterion for convergence of series of real numbers.

Let $\sum_{n \geq 0} x_n$ and $\sum_{n \geq 0} y_n$ be two series of real numbers. The product series $\sum_{n \geq 0} x_n y_n$ is also convergent provided that one of the following two conditions is satisfied:

(A): $(x_n)_{n \geq 0}$ is decreasing to zero and $\exists N > 0$ such that for all $n \geq 0$, $\left| \sum_{i=0}^{n} y_i \right| \leq N$.

(B): $(x_n)_{n \geq 0}$ is monotone and bounded and the series $\sum_{n \geq 0} y_n$ is convergent.

Application.

Study the nature of the following series:

$$\sum_{n \geq 1} \frac{\cos n \cdot \cos \frac{1}{n}}{n}$$

Solution.

We write $\frac{\cos n \cdot \cos \frac{1}{n}}{n} = u_n \cdot v_n$, where
\[ u_n = \cos \frac{1}{n}, \text{ and } v_n = \frac{\cos n}{n}. \text{ Now, the series } \sum_{n=1} v_n \text{ is convergent, and the sequence } (u_n)_{n=1} \text{ is monotonic and bounded. By Abel-Dirichlet criterion, the series } \sum_{n=1} u_n v_n \text{ is convergent.} \]

**Another application.**

Study the nature of the following series,

\[ \sum_{n=1} \frac{\cos(nx)}{n^2}, \text{ where } x \in \mathbb{R} \]

and \( x \in (0, \infty). \)

**Solution.**

If there exists \( k \in \mathbb{Z} \) such that \( x = 2k\pi \), then the series is \( \sum_{n=1} \frac{1}{n^2} \) which is the well-known harmonic series. If \( x \neq 2k\pi \) for any \( k \in \mathbb{Z} \), then we have \( \frac{\cos(nx)}{n^2} = x_n \cdot y_n \), where \( x_n = \frac{4}{n^2}, y_n = \cos(n) \).

Clearly, \( x_n \to 0 \) and \( x_n \) is decreasing. Moreover,

\[ \left| \sum_{k=1}^{\infty} y_k \right| \leq \frac{1}{|\sin(n \pi/2)|}, \text{ for all } n \geq 1. \text{ By the Abel-Dirichlet criterion, it follows that our series is convergent.} \]
Problem.

Let \((a_n)_n\) be a sequence such that the series \(\sum_{n=1}^{\infty} a_n\) converges to \(A\).

(a) Show that any regrouping of the terms

\[(a_1 + \cdots + a_n) + (a_{n+1} + \cdots + a_m) + (a_{m+1} + \cdots + a_{n_2}) + \cdots\]

gives a series that also converges to \(A\).

(b) Give an example of a sequence \((a_n)_n\) such that the series

\[\sum_{k=1}^{\infty} (a_{2k-1} + a_{2k})\]

converges but \(\sum_{n=1}^{\infty} a_n\) diverges.

Solution.

(a) Let \(S_n\) denote the sequence of partial sums for the original series, and note that the sequence of partial sums for the new series is just the subsequence \((S_{2k})_{k=1}^{\infty}\). Since the sequence \(S_k \to A\), it follows that the subsequence \((S_{2k})_{k=1}^{\infty}\) also converges to \(A\).
and we are done.

(b) Let \( a_n = (-1)^n \). Clearly, the series \( \sum_{n=1}^{\infty} a_n \) diverges but after regrouping since \( b_k = a_{2k-1} + a_k = 0 \), the new series \( \sum_{k=1}^{\infty} b_k = 0 \) converges. \( \square \)

Problem.

Show that if the series \( \sum_{n=1}^{\infty} |a_{n+1} - a_n| \) converges, then the sequence \( (a_n) \) converges. Give an example of a convergence sequence \( (a_n) \) so that \( \sum_{n=1}^{\infty} |a_{n+1} - a_n| \) diverges.

Solution.

It is sufficient to show that the sequence \( (a_n) \) is Cauchy. We have

\[ |a_m - a_n| \leq |a_{m+1} - a_n| + |a_{m+2} - a_{m+1}| + \cdots + |a_n - a_{m+1}|. \]

Now, take \( \varepsilon > 0 \). Then by Cauchy criterion applied to the series \( \sum_{n=1}^{\infty} |a_{n+1} - a_n| \), there is \( N \) such that

\[ \sum_{k=n}^{m} |a_{k+1} - a_k| = |a_{m+1} - a_n| + \cdots + |a_{m} - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| < \varepsilon, \]

for all \( m > n \geq N \). Consequently, \( |a_m - a_n| < \varepsilon \) for all \( m > n \geq N \).

An example of such a sequence is

\[ a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ -\frac{1}{n} & \text{if } n \text{ is even} \end{cases} \]

\( \square \)