Recitation Week 3. (Introduction to Math Analysis - 0420)

Firstly, let us remember some remarkable limits from Calculus I:

1) \[ \lim_{\mu \to 0} \frac{\sin \mu}{\mu} = 1 \]
2) \[ \lim_{\mu \to \infty} \left( 1 + \frac{1}{\mu} \right)^\mu = e \]
3) \[ \lim_{\mu \to 0} (1 + \mu)^\mu = e \]
4) \[ \lim_{\mu \to 0} \frac{\arcsin \mu}{\mu} = 1 \]
5) \[ \lim_{\mu \to 0} \frac{\tan \mu}{\mu} = 1 \]
6) \[ \lim_{\mu \to 0} \frac{\log(1+\mu)}{\mu} = 1 \]
7) \[ \lim_{\mu \to 0} \frac{(1+\alpha)^\mu - 1}{\mu} = \alpha. \]

Exc.

Prove that the following limits do not exist:

\[ \lim_{x \to 0} \frac{1}{x} , \lim_{x \to \infty} e^{x \cos \frac{1}{x}} , \]

\[ \lim_{x \to 0} e^{-\frac{1}{x}}. \]

Solution.

Since \[ \lim_{n \to \infty} \frac{1}{n} = 1 \] and \[ \lim_{n \to \infty} \frac{1}{n} = -1, \] this implies that \[ \lim_{x \to 0} \frac{1}{x} \] does not exist.

For the second one, let us consider the following two limits:

\[ \lim_{n \to \infty} e^{\frac{1}{2n} \cdot \cos \frac{1}{2n}} = 1 \] and \[ \lim_{n \to \infty} e^{\frac{1}{(2n+1)^2} \cdot \cos \frac{1}{1/(2n+1)}} = 1 \]

The last one goes along similar lines. Indeed, we have \[ \lim_{n \to \infty} e^{-\frac{1}{n}} = \lim_{n \to \infty} e^{-n} = 0 \] and \[ \lim_{n \to \infty} e^{-\frac{1}{n}} = \lim_{n \to \infty} e^{-n} = \infty \] and we are done. \( \square \)
Problem.
Prove that if \( \lim_{x \to \infty} f(x) = \infty \) and \( g(x) \geq 0 \) on some set \( (a, \infty) \), where \( a \in \mathbb{R} \), then \( \lim_{x \to \infty} f(x)g(x) = \infty \).

Solution.
Let \( M > 0 \). Since \( \lim_{x \to \infty} f(x) = \infty \), for \( M_1 = \frac{M}{2} \), there exists \( K_1 > 0 \) such that \( f(x) > \frac{M}{2} \) whenever \( x > K_1 \). Now, set \( K = \max(K_1, a) \). Then when \( x > K \),

\[
f(x)g(x) \geq \frac{M}{2} \quad x = M.
\]

Hence \( \lim_{x \to \infty} f(x)g(x) = \infty \). □

Problem.
Let \( S \subseteq \mathbb{R} \) and \( c \) be a cluster point of \( S \). Let \( f: S \to \mathbb{R} \) be a function. And suppose that the limit of \( f(x) \) as \( x \) goes to \( c \) exists. Suppose that there are two real numbers \( a \) and \( b \) such that

\[
a \leq f(x) \leq b \quad \text{for all } x \in S.
\]

Show that

\[
a \leq \lim_{x \to c} f(x) \leq b.
\]

Solution.
Let \( (x_n)_{n \geq 1} \) be a sequence of numbers from \( S \setminus \{c\} \) such that \( x_n \to c \). We know that \( a \leq f(x_n) \leq b \), for all \( n \). Now, by applying Corollary 2.2.4. it follows that
\[ a \leq \lim_{n \to \infty} f(n) \leq b. \]

Since the limit of \( f(x) \) exists at \( c \) and \( x_n \to c \), we know that \( L = \lim_{x \to c} f(x) \).

Finally, this implies that \( a \leq \lim_{x \to c} f(x) \leq b \).

**Problem.**

Find the limits or show that they do not exist:

a) \( \lim_{x \to 0} \frac{x^2 \cos \frac{1}{x}}{x} \);  
b) \( \lim_{x \to 0} \sin x \cos \frac{1}{x} \);

**Solution.** In both cases we shall show that the limits are zero.

a) Indeed, we have

\[ 0 \leq \left| \frac{x^2 \cos \frac{1}{x}}{x} \right| \leq \left| x^2 \cdot 1 \right| = |x^2| \text{ or in more detail, we can write} \]

\[ 0 \leq \left| \frac{x^2 \cos \frac{1}{x}}{x} \right| = |x^2| \cdot \left| \cos \frac{1}{x} \right| \leq |x^2| \]

By the squeeze theorem, we have \( \lim_{x \to 0} \frac{x^2 \cos \frac{1}{x}}{x} = 0 \).
b) In the same fashion, we have

\[ 0 \leq |\sin x \cos \left( \frac{1}{x} \right)| = |\sin x| \cdot |\cos \frac{1}{x}| \leq |\sin x| \leq |x| \leq 1 \]

Problem.

Let \( A \subseteq S \), show that if \( c \) is a cluster point of \( A \), then \( c \) is a cluster point of \( S \).

Solution.

Let \( c \) be a cluster point of \( A \). By the definition of cluster points, we have

\[ \forall \varepsilon > 0, \quad (c - \varepsilon, c + \varepsilon) \cap A \setminus \{c\} \neq \emptyset. \]

Since \( A \subseteq S \), then

\[ (c - \varepsilon, c + \varepsilon) \cap A \setminus \{c\} \subseteq (c - \varepsilon, c + \varepsilon) \cap S \setminus \{c\}. \]

So,

\[ (c - \varepsilon, c + \varepsilon) \cap S \setminus \{c\} \neq \emptyset. \]

This implies that \( c \) is a cluster point of \( S \).
Problem.
Prove that if \( \lim_{x \to a} f(x) = 0 \) and \( g \) is bounded on some neighborhood when \( x = a \), then \( \lim_{x \to a} f(x)g(x) = 0 \).

Solution.
Since \( g \) is bounded on a neighborhood of \( x = a \), it follows that \( |g(x)| \leq M \) for \( x \) such that \( |x - a| < \delta \). Now, let \( \varepsilon > 0 \). Then, there exists \( \delta_2 > 0 \) such that \( |f(x)| < \frac{\varepsilon}{M} \) whenever \( 0 < |x - a| < \delta_2 \). Now, we set \( \delta = \min(\delta_1, \delta_2) \). Then when \( 0 < |x - a| < \delta \), we obtain

\[
|f(x)g(x)| = |f(x)| |g(x)| < \frac{\varepsilon}{M} \cdot M = \varepsilon.
\]

Hence \( \lim_{x \to a} f(x)g(x) = 0 \). \( \square \)