Problem 1. Determine which of the following sets are rings:

a) \( S = \{a + b\sqrt{3}/a, b \in \mathbb{Z}\} \);

b) \( S = \{a + bi/a, b \in \mathbb{Q}\} \);

c) \( S = \{\begin{bmatrix} a & b \\ -b & a \end{bmatrix}/a, b \in \mathbb{R}\} \);

d) \( S = \{A \in M(2, \mathbb{R})/ \det(A) = 0\} \).

Problem 2. Show that the set of all functions \( f : \mathbb{R} \to \mathbb{R} \), denoted by \( F(\mathbb{R}) \), forms a ring under the operations of pointwise addition and multiplication.

Problem 3. Let \( R \) be a ring. The center of \( R \) is defined as follows:

\[ Z(R) = \{x \in R/xy = yx, \forall y \in R\} . \]

Show that \( Z(R) \) is a subring of \( R \).

Solution. Since for all \( y \in R \), \( y0=0y=0 \), it follows that \( 0 \in Z(R) \), and thus \( Z(R) \) is nonempty. Now, let \( u, v \in Z(R) \) be arbitrary. Then for all \( y \in R \), we have \( yu = uy \) and \( yv = vy \). So

\[ (u - v)y = uy - vy = yu - yv = y(u - v), \]

and thus \( u - v \in Z(R) \). Also,

\[ (uv)y = u(vy) = u(yv) = (uy)v = (yu)v = y(vu), \]

and thus \( uv \in Z(R) \). By the subring test, \( Z(R) \) is a subring of \( R \). \( \square \)

Problem 4. Let \( R \) be a ring. Show that \( (a + b)^2 = a^2 + 2ab + b^2 \) if and only if \( R \) is commutative.

Solution. Indeed, we have

\[ (a + b)^2 = (a + b)(a + b) = aa + ab + ba + bb = a^2 + ab + ba + b^2. \]

If \( ab = ba \), then we have \( (a + b)^2 = a^2 + 2ab + b^2 \). For the other implication, we know that

\[ (a + b)^2 = a^2 + 2ab + b^2 = a^2 + ab + ba + b^2. \]

This implies that \( ab + ba = 2ab \) or equivalently \( ab = ba \). \( \square \)

Problem 5. Find all the zero divisors in the following rings:
Problem 6.  a) Give an example of a commutative ring with no zero divisors that is not an integral domain;
   b) Give an example of a ring with unity and no zero divisors that is not an integral domain.

Solution. a) $2\mathbb{Z}$ (the set of even integers).
   b) $\mathbb{Z}_6$ or $\mathbb{Z}_n$ when $n$ is not a prime.

Problem 7. Show that the following rings are integral domains:

a) $\mathbb{Q}(i) = \{a + bi/a, b \in \mathbb{Q}\};$
   b) $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5}/a, b \in \mathbb{Q}\}.$

Problem 8. Find all the units in the following rings:

a) $\mathbb{Z}_{10};$
   b) $\mathbb{Z}[i];$
   c) $\mathbb{Q}(\sqrt{3}).$

Problem 9. a) Find all the idempotent elements in $\mathbb{Z}_{12};$
   b) Find all the nilpotent elements in $\mathbb{Z}_{24}.$
(An element $a$ in a ring $R$ is called *nilpotent* if for some $k \geq 1$ we have $a^k = 0.$)

Solution. a) Note that an element $a$ is an *idempotent* in a ring $R$ if $a^2 = a,$ for all $a \in R.$ In $\mathbb{Z}_{12},$ we have $0^2 = 0, 1^1 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 4, 5^2 = 1, 6^2 = 0, 7^2 = 1, 8^2 = 4, 9^2 = 9, 10^2 = 4, 11^2 = 1.$ Therefore the idempotents are $0, 1, 4, 9.$
   b) We are looking for elements which have power divisible by 24. Since $24 = 2^3 \cdot 3,$ then any element which does not have both a factor of 2 and a factor of 3 is dropped out! On the other hand, one can easily check that the set of nilpotent elements is given by $\{0, 6, 12, 18\}.$ Indeed, we have $0^1 = 0$ divisible by 24, $6^3 = 216 = 24 \cdot 9,$ $12^2 = 144 = 24 \cdot 6,$ and $18^3 = 5832 = 243 \cdot 24.$\)

Problem 10. Let $F$ be a field with char $F = p > 0.$ Show that for any elements $a, b \in F,$ we have

$$(a + b)^p = a^p + b^p.$$  

Solution. By the binomial expansion formula, we have

$$(a + b)^p = \binom{p}{0}a^pb^0 + \binom{p}{1}a^{p-1}b^1 + \ldots + \binom{p}{k}a^{p-k}b^k + \ldots + \binom{p}{p-1}a^1b^{p-1} + \binom{p}{p}a^0b^p.$$
In other words, the above equality can be rewritten as

\[(a + b)^p = a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k b^{p-k} + b^p,\]

where \(\binom{p}{k} = \frac{p!}{k!(p-k)!}\). On the other hand, we know that

\[\binom{p}{k} = \frac{(p-k+1) \ldots (p-1)p}{k!} \equiv 0 (\text{mod } p),\]

and thus the sum \(\sum_{k=1}^{p-1} \binom{p}{k} a^k b^{p-k}\) vanished in \(\text{char}(R) = p > 0\). Therefore, \((a + b)^p = a^p + b^p\) and we are done. □