Problem 1. Let $A \in M_2(\mathbb{R})$ such that $\det A = -1$. Show that $\det(A^2 + I_2) \geq 4$.

Romanian National Olympiad SL, 2003

Problem 2. Let $A, B \in M_2(\mathbb{R})$. Show that

$$\det(A^2 + B^2) \geq \det(AB - BA).$$

Problem 3. Let $A, B \in M_2(\mathbb{R})$ such that $AB = BA$ and $\det(A^2 + B^2) = 0$. Show that $\det A = \det B$.

Problem 4. Let $A, B \in M_2(\mathbb{Z})$ such that $A, A + B, A + 2B, A + 3B$ and $A + 4B$ are invertible and moreover their inverses are in the ring $M_2(\mathbb{Z})$. Show that the matrix $A + 5B$ is invertible and $(A + 5B)^{-1} \in M_2(\mathbb{Z})$.

Putnam A4, 1994

Problem 5. Let $S$ be the set of all $2 \times 2$ matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

whose entries $a, b, c, d$ (in that order) form an arithmetic progression. Find all matrices $M$ in $S$ for which there is some integer $k > 1$ such that $M^k$ is also in $S$.

Putnam B3, 2015

Problem 6. Let $A \in M_2(\mathbb{R})$ such that $\text{tr}(A) > 2$. Show that $A^n \neq I_2$ for all $n \geq 1$.

Romanian National Olympiad, 1988

Problem 7. Are there any matrices $A, B \in M_3(\mathbb{C})$ such that $(AB - BA)^2 = I_3$?

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Problem 8. Let $A, B \in M_3(\mathbb{Z})$ such that $\det(A) = \det(B) = 1$. Show that $\det(A + \sqrt{2}B) \neq 0$.

Problem 9. Let $A \in M_3(\mathbb{C})$ such that $\text{tr}(A) = \text{tr}(A^2) = 0$. Show that if $|\det A| < 1$, then the matrix $I_3 + A^n$ is invertible for all $n \geq 1$.

Problem 10. Let $A, B \in M_3(\mathbb{C})$ such that $(AB)^2 = A^2B^2$ and $(BA)^2 = B^2A^2$. Prove that $(AB - BA)^3 = O_3$.


Romanian National Olympiad SL, 2014