Problem 1. Calculate the quotients and remainders on division of the indicated \( f(x) \) by \( g(x) \) in the indicated polynomial rings:

a) \( f(x) = 4x^3 - 2x^2 + 5x - 3 \), \( g(x) = x^2 + x + 1 \) in \( \mathbb{Q}[x] \)

b) \( f(x) = 3x^4 + 2x + 3 \), \( g(x) = x^2 + 2x + 1 \) in \( \mathbb{Z}_5[x] \)

c) \( f(x) = x^4 + x^3 + x^2 + x + 1 \), \( g(x) = x + 1 \) in \( \mathbb{Z}_2[x] \)

d) \( f(x) = x^4 + 9x^2 + 5 \), \( g(x) = x^2 + 3x + 4 \) in \( \mathbb{Z}_{11}[x] \).

Problem 2. Find all polynomials of degree 2 in \( \mathbb{Z}_2[x] \) and in \( \mathbb{Z}_3[x] \).

Problem 3. Find all zeros of \( f(x) = x^2 - 1 \) in \( \mathbb{Z}_{15} \). Does this contradict a theorem about the degree of a polynomial and its number of roots?

Problem 4. Calculate \( \text{gcd}(f(x), g(x)) \) for the indicated \( f(x) \) and \( g(x) \) in the indicated polynomial rings \( F[x] \). Also, in each case find \( u(x) \) and \( v(x) \) such that \( \text{gcd}(f(x), g(x)) = u(x)f(x) + v(x)g(x) \),

a) \( f(x) = x^4 - x^2 - 2 \), \( g(x) = x^3 + x^2 + x + 1 \) in \( \mathbb{Q}[x] \)

b) \( f(x) = x^4 + x^3 + x + 1 \), \( g(x) = x + 1 \) in \( \mathbb{Z}_2[x] \)

c) \( f(x) = x^3 + 1 \), \( g(x) = x + 2 \) in \( \mathbb{Z}_5[x] \)

d) \( f(x) = x^3 + 2x + 1 \), \( g(x) = x + 2 \) in \( \mathbb{Z}_9[x] \).

Problem 5. Find the zeros of the indicated \( f(x) \) in the indicated field:

a) \( f(x) = x^2 + x + 1 \) in \( \mathbb{Z}_3 \)

b) \( f(x) = x^3 + x^2 + x + 1 \) in \( \mathbb{R} \)

c) \( f(x) = x^3 + x^2 + x + 1 \) in \( \mathbb{C} \)

d) \( f(x) = x^8 - 1 \) in \( \mathbb{R} \).

Problem 6. Show that

(i) \( f(x) = x^3 + 2x + 1 \) is irreducible over \( \mathbb{Z}_5 \)

(ii) \( f(x) = x^3 + x + 1 \) is irreducible over \( \mathbb{Z}_7 \)

(iii) \( f(x) = x^4 - 2 \) is irreducible over \( \mathbb{Q} \) but reducible over \( \mathbb{R} \)

(iv) \( f(x) = x^4 - 2x^2 + 4 \) is irreducible over \( \mathbb{Q} \).

Problem 7. Using any criteria given in class, determine whether the indicated polynomial \( f(x) \) in \( \mathbb{Z}[x] \) is reducible over \( \mathbb{Q} \). Justify your answers.

a) \( f(x) = 10x^7 - 6x^4 + 15x^2 + 18x - 6 \)
b) \( f(x) = x^4 - 4x^2 + 4x - 1 \)
c) \( f(x) = 3x^4 + 5x + 1 \)
d) \( f(x) = x^4 + 4. \)

**Problem 8.** Let \( f(x) = x^{n-1} + x^{n-2} + \ldots + x + 1 \in \mathbb{Q}[x], \) where \( n \) is not a prime. Show that \( f(x) \) is not irreducible over \( \mathbb{Q}. \)

**Problem 9.** Describe the elements of \( \mathbb{Q}[x]/\langle x^2 - 3 \rangle, \) and show that this quotient ring of \( \mathbb{Q}[x] \) isomorphic to \( \mathbb{Q}(\sqrt{3}). \)

**Problem 10.** Show that \( \mathbb{R}[x]/\langle x^2 + 1 \rangle \) is isomorphic to \( \mathbb{C}. \)