Problem 1. Find the following limits:

(i) \[ \lim_{n \to \infty} n \left( \log 2 - \sum_{k=1}^{n} \frac{1}{n+k} \right) \]

(ii) \[ \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} \sqrt{1 + \frac{1}{n+k}} \right)^n \]

Problem 2. Let \((a_n)_{n \geq 1}\) be a sequence of positive integers such that \(\sum_{n \geq 1} a_n^3\) converges. Show that the series \(\sum_{n \geq 1} \frac{a_n}{n}\) converges also.

Problem 3. Let \((a_n)_{n \geq 1}\) be a sequence of positive reals such that the series \(\sum_{n=1}^{\infty} a_n\) is convergent. Show that the series \(\sum_{n=1}^{\infty} \frac{a_n^2}{n+1}\) also converges.

Putnam B5, 1988

Problem 4. Let \(k\) be an integer greater than 1. Suppose that \(a_0 > 0\), and define the sequence \((a_n)_{n \geq 0}\),

\[ a_{n+1} = a_n + \frac{1}{\sqrt[n]{a_n}}, n \geq 0. \]

Evaluate

\[ \lim_{n \to \infty} \frac{a_{n+1}}{n^k}. \]

Putnam B6, 2006
Problem 5. Let \( f(x) = \sum_{k=1}^{n} a_k \sin(kx) \), with \( a_1, a_2, \ldots, a_n \in \mathbb{R} \), \( n \geq 1 \). Prove that if \( f(x) \leq |\sin x| \), for all \( x \in \mathbb{R} \), then
\[
\left| \sum_{k=1}^{n} ka_k \right| \leq 1.
\]

Putnam A1, 1967

Problem 6. Let \( f : [a, b] \to \mathbb{R} \) a function, continuous on \([a, b]\), and twice differentiable on \((a, b)\). If \( f(a) = f(b) \) and \( f'(a) = f'(b) \), prove that for every real number \( \lambda \), the equation
\[
f''(x) - \lambda(f'(x))^2 = 0
\]
has at least one solution in the interval \((a, b)\).

Problem 7. Does there exist a continuously differentiable function \( f : \mathbb{R} \to (0, \infty) \) such that \( f'(x) = f(f(x)) \) for all \( x \)?

International Mathematics Competition, 2002

Problem 8. Let \( f \) be a three times differentiable function (defined on \( \mathbb{R} \) and real-valued) such that \( f \) has at least five distinct real zeros. Prove that \( f + 6f' + 12f'' + 8f''' \) has at least two distinct real zeros.

Putnam B1, 2015

Problem 9. For each continuous function \( f : [0, 1] \to \mathbb{R} \), let \( I(f) = \int_{0}^{1} x^2f(x) \, dx \) and \( J(f) = \int_{0}^{1} x(f(x))^2 \, dx \). Find the maximum value of \( I(f) - J(f) \) over all such functions \( f \).

Putnam B5, 2006

Problem 10. Find all continuous functions \( f : \mathbb{R} \to [1, \infty) \) for which there exists \( a \in \mathbb{R} \) and \( k \) positive integer such that
\[
f(x)f(2x)\ldots f(nx) \leq an^k,
\]
for every real number \( x \) and positive integer \( n \).

Romanian National Olympiad, 1999

Problem 11. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function such that \( \int_{0}^{1} f(x) \, dx = 0 \).

Show that there exists \( c \in (0, 1) \) such that
\[
f(c) = \int_{0}^{c} f(x) \, dx.
\]
Problem 12. Find

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n \frac{x \log(1 + x/n)}{1 + x} \, dx.$$  

American Mathematical Monthly, 2006

Problem 13. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function, differentiable on \((0, 1)\), with the property that there exists \( \alpha \in (0, 1) \) such that \( \int_0^\alpha f(x) \, dx = 0 \). Prove that

$$\left| \int_0^1 f(x) \, dx \right| \leq \frac{1 - \alpha}{2} \sup_{x \in (0, 1)} |f'(x)|.$$  

Romanian National Olympiad, 1983

Problem 14. Let \( f : [0, 1] \to \mathbb{R} \) be an integrable function such that

$$\int_0^1 f(x) \, dx = \int_0^1 x f(x) \, dx = 1.$$  

Show that

$$\int_0^1 f^2(x) \, dx \geq 4.$$  

Romanian National Olympiad, 2004

Problem 15. Evaluate

$$\int_0^1 \frac{\log(1 + x)}{1 + x^2} \, dx.$$  

Putnam A5, 2005

Problem 16. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be continuous functions such that \( f(x+1) = f(x) \) and \( g(x+1) = g(x) \) for all real numbers \( x \). Prove that

$$\lim_{n \to \infty} \int_0^1 f(x) g(nx) \, dx = \int_0^1 f(x) \, dx \int_0^1 g(x) \, dx.$$  

Putnam B3, 1967