LECTURE NOTES

MATH 2360: LINEAR ALGEBRA

CHAPTER I:
SYSTEMS OF LINEAR EQUATIONS
Lecture 1. Introduction to systems of linear equations.

- recognize a linear equation in n variables
- find a parametric representation of a solution set
- determine whether a system of linear equations is consistent or inconsistent
- use back-substitution and Gaussian elimination to solve a system of linear equations.

I. Linear equations in n variables.

Recall from analytic geometry that the equation of a line in two-dimensional space has the form

\[ a_1 x + a_2 y = b, \quad a_1, a_2, b \text{ constants} \]

We call this equation a linear equation in two variables.

Similarly, the equation of a plane in 3-dimensional space has the form

\[ a_1 x + a_2 y + a_3 z = b, \quad a_1, a_2, a_3, b \text{ constants} \]

We call this a linear equation in 3-variables.
Definition.

A linear equation in $n$ variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

The coefficients $a_1, a_2, \ldots, a_n$ are real numbers, and the constant term $b$ is a real number. The number $a_1$ is the leading coefficient, and $x_1$ is the leading variable.

Examples:

a) $2x + 3y = 7$ (linear equation)
b) $(\sin x)x_1 - 4x_2 = e^2$ (linear equation)
c) $e^x - 2y = 4$ (not a linear equation)
d) $\sin x_1 + 2x_2 - 3x_3 = 0$ (not a linear equation)

(2) Solutions and solution sets.

A solution of a linear equation in $n$ variables is a sequence of $n$ real numbers $s_1, s_2, s_3, \ldots, s_n$ that satisfy the equation when you substitute the values $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ into the equation.
Example.

$x_1 = 2, x_2 = 1$ satisfy the equation $x_1 + 2x_2 = 4$.

The set of all solutions is a solution set, and when you found this set, you have solved the equation. To describe the entire solution set of a linear equation, use a parametric representation as we shall illustrate in the following examples.

Example 1.

Solve the linear equation $x_1 + 2x_2 = 4$.

One can solve this equation in one variable $x_1 = 4 - 2x_2$. In this form $x_2$ is free. By letting $x_2 = t$, you can represent the solution set as

$x_1 = 4 - 2t, \quad x_2 = t, \quad t$ any real variable.

To obtain particular solutions, assign values to the parameter $t$. For instance, $t = 1$ yields the solution $x_1 = 2$ and $x_2 = 1$.

Example 2.

Solve the linear equation $3x + 2y - 2 = 3$.

Choosing $y$ and $z$ to be free variables, solve for $x$ and we obtain.
\[ 3x = 3 - 2y + 2 \Rightarrow x = 1 - \frac{2}{3}y + \frac{1}{3}z. \]

Letting \( y = s \) and \( z = t \), you obtain the parametric representation

\[ x = 1 - \frac{2}{3}s + \frac{1}{3}t, \quad y = s, \quad z = t, \]

where \( s, t \) are any real numbers. Two particular solutions are

\[ x = 4, \quad y = 0, \quad z = 0 \quad \text{and} \quad x = 1, \quad y = 1, \quad z = 2. \]

Example 3.

Solve the linear equation \( x + y + z = 1 \).

Again, choosing \( y \) and \( z \) to be free variables, solve for \( x \) to obtain

\[ x = 1 - y - z. \]

Letting \( y = s \) and \( z = t \), we obtain the following parametric representation

\[ x = 1 - s - t, \quad \text{where} \quad s, t \in \mathbb{R}. \]


A system of \( n \) linear equations in \( n \) variables is a set of \( n \) equations, each of which in the same \( n \) variables:
\[
\begin{align*}
\begin{cases}
\begin{align*}
 a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
 & \\
 a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\end{cases}
\end{align*}
\]

A system of linear equations is also called a *linear system*. A *solution* of a linear system is a sequence of numbers \(s_1, s_2, \ldots, s_n\) that is a solution of each equation in the system. For example, the system

\[
\begin{cases}
3x_1 + 2x_2 = 3 \\
-x_1 + x_2 = 4
\end{cases}
\]

has \(x_1 = -1\) and \(x_2 = 3\) as solution because \(x_1 = -1\) and \(x_2 = 3\) satisfy both equations. On the other hand, \(x_1 = 1\) and \(x_2 = 0\) is not a solution of the system because these values satisfy only the first equation of the system.

Q: How to solve the system of two equations?

- graph the two lines \(\begin{cases} 3x - y = 1 \\
2x - y = 0 \end{cases}\) in the \(xy\)-plane. Where do they intersect?
How many solutions does this system of linear equations have?

* Repeat this analysis for the pairs of lines
  \[
  \begin{align*}
  3x - y &= 1 \\
  2x - y &= 0 \\
  \end{align*}
  \quad \text{and} \quad 
  \begin{align*}
  3x - y &= 1 \\
  6x - 2y &= 2 \\
  \end{align*}
  \]

* What basic type of solution sets are possible for a system of two linear equations in two variables?

Example 4.

Solve and graph each system of linear equations:

a) \( \begin{align*}
  x + y &= 3 \\
  x - y &= -1 \\
  \end{align*} \)

b) \( \begin{align*}
  x + y &= 3 \\
  2x + 2y &= 6 \\
  \end{align*} \)

c) \( \begin{align*}
  x + y &= 3 \\
  x + y &= 1 \\
  \end{align*} \)

a) The system has exactly one solution, \( x = 1 \) and \( y = 2 \). One way to obtain the solution is to add the two equations to give \( 2x = 2 \), which implies \( x = 1 \) and so \( y = 2 \). The graph of this system is two intersecting lines.

\[ \text{Graph of intersecting lines} \]
b) This system has infinitely many solutions because the second equation is the result of multiplying both sides of the first equation by 2. A parametric representation of the solution set is

\[ x = 3 - t, \quad y = t, \quad t \in \mathbb{R}. \]

The graph of this system is given by

![Graph of the system](image)

c) This system has no solution because the sum of two numbers cannot be 3 and 1 at the same time. The graph of this system is two parallel lines, as shown in the graph below.

![Graph of the system](image)
For a system of linear equations, precisely one of the statements below is true:

1. The system has exactly one solution (consistent system)
2. The system has infinitely many solutions (consistent system)
3. The system has no solution (inconsistent system)

4. **Solving a system of linear equations.**
   Which system is easier to solve algebraically?

\[
\begin{cases}
  x - 2y + 3z = 9 \\
  -x + 3y = -4 \\
  2x - 5y + 5z = 17
\end{cases}
\]

\[
\begin{cases}
  x - 2y + 3z = 9 \\
  y + 3z = 5 \\
  z = 2
\end{cases}
\]

To solve these systems we use **back-substitution.**

**Example 5.** Solve the system

\[
\begin{cases}
  x - 2y = 5 \text{ equation 1} \\
  y = -2 \text{ equation 2}
\end{cases}
\]

\[
x - 2(-2) = 5 \Rightarrow x = 5 + 4 = 9.
\]

\[
y = -2.
\]
Example 6. Solve the system

\[
\begin{align*}
  x - 2y + 3z &= 9 \quad \text{Equation 1} \\
  y + 3z &= 5 \quad \text{Equation 2} \\
  z &= 2 \quad \text{Equation 3}
\end{align*}
\]

Solution.

From Equation 3, you know the value of \( z \).
Now, to solve it for \( y \), we plug in \( z = 2 \) into Equation 2 to obtain

\[
y + 3 \cdot 2 = 5 \implies y = -1.
\]

Next, we substitute \( y = -1 \) and \( z = 2 \) into Equation 1 and we obtain \( x = 1 \).

Operations that produce equivalent systems.

Each of these operations on a system of linear equations produces an equivalent system:

- interchange two equations;
- multiply an equation by a nonzero constant;
- add a multiple of an equation to another equation.

Rewriting a system of linear equations in row-echelon form involves...
a chain of equivalent systems, using one of the three basic operations to obtain each system. This is called Gaussian elimination, after the famous German mathematician Carl Friedrich Gauss (1777–1855).

Example 7. Using elimination to rewrite a system in row-echelon form:

Solve the system:

\[
\begin{align*}
X - 2Y + 3Z &= 9 & \quad \text{Equation 1} \\
-X + 3Y &= -4 & \quad \text{Equation 2} \\
2X - 5Y + 5Z &= 17 & \quad \text{Equation 3}
\end{align*}
\]

Solution.

\[
\begin{align*}
X - 2Y + 3Z &= 9 \\
Y + 3Z &= 5 \quad \text{add Equation 1 to Equation 2 to produce a new Equation 2} \\
2X - 5Y + 5Z &= 17
\end{align*}
\]

\[
\begin{align*}
X - 2Y + 3Z &= 9 \\
Y + 3Z &= 5 \quad \text{add \(-2\) times Equation 1 to Equation 3 to produce a new Equation 3} \\
-Y + Z &= -1
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
-x + 2y + 3z &= 9 \\
y + 3z &= 5 \\
2z &= 4 
\end{cases} & \text{ add Equation 2 to Equation 3 produces a new Equation 3} \\
\begin{cases}
-x + 2y + 3z &= 9 \\
y + 3z &= 5 \\
2z &= 2 
\end{cases} & \text{ multiply Equation 3 by } \frac{1}{2} \text{ to produce a new Equation 3} \\
\text{Using back-substitution, we obtain the unique solution of the system } (x, y, z) = (1, -1, 2). \quad \square
\end{align*}
\]

Example 8. An inconsistent system

Solve the system

\[
\begin{align*}
\begin{cases}
x_1 - 3x_2 + x_3 &= 1 \\
2x_1 - x_2 - 2x_3 &= 2 \\
x_1 + 2x_2 - 3x_3 &= -1 
\end{cases}
\end{align*}
\]

Solution,

\[
\begin{align*}
\begin{cases}
x_1 - 3x_2 + x_3 &= 1 \\
5x_2 - 4x_3 &= 0 \leftarrow \text{ add } -2 \text{ times Equation 1 to Equation 2 to produce a new Equation 2} \\
x_1 + 2x_2 - 3x_3 &= -1
\end{cases}
\end{align*}
\]
\[ \begin{align*}
&x_1 - 3x_2 + x_3 = 1 \\
&5x_2 - 4x_3 = 0 \\
&5x_2 - 4x_3 = -2
\end{align*} \]

Add $-1$ times Equation 1 to Equation 3 to produce a new Equation 3.

\[ \begin{align*}
&x_1 - 3x_2 + x_3 = 1 \\
&5x_2 - 4x_3 = 0 \\
&0 = -2
\end{align*} \]

Subtract Equation 2 from Equation 3 to produce a new Equation 3.

**Lecture 2. Gaussian Elimination and Gauss-Jordan Elimination.**

- determine the size of a matrix and write an augmented or coefficient matrix from a system of linear equations
- use matrices and Gaussian elimination with back-substitution to solve a system of linear equations
- solve a homogeneous system of linear equations.

1) **Matrices.**

We begin with the definition of a matrix.
If \( m \) and \( n \) are positive integers, then \( m \times n \) (read "m by n") matrix is a rectangular array

\[
A = (a_{ij}) = \left[ \begin{array}{cccc}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{array} \right]
\]

in which each entry, \( a_{ij} \), of the matrix is a number. An \( m \times n \) matrix has \( m \) rows and \( n \) columns. Matrices are usually denoted by capital letters.

The entry \( a_{ij} \) is located in the \( i \)th row and \( j \)th column. The index \( i \) is called the row subscript because it identifies the row in which the entry lies, and the index \( j \) is called the column subscript because it identifies the column in which the entry lies.

A matrix with \( m \) rows and \( n \) columns is of size \( m \times n \). When \( m = n \), the matrix is square of order \( n \) and the entries \( a_{11}, a_{22}, a_{33}, \ldots, a_{nn} \) are the main diagonal entries.
Example 1.

a) \([2]\) is a matrix of size \(1 \times 1\)

b) \([\begin{bmatrix} 0 & 0 \end{bmatrix}]\) is a matrix of size \(2 \times 2\)

c) \([\begin{bmatrix} 1 & 2 & -7 \\ \pi & \sqrt{2} & 4 \end{bmatrix}]\) is a matrix of size \(2 \times 3\).

One common use of matrices is to represent systems of linear equations. The matrix derived from the coefficients and constant terms of a system of linear equations is the **augmented matrix** of the system. The matrix containing only the coefficients of the system is the **coefficient matrix** of the system. Here is an example:

System

\[
\begin{cases}
X - 4Y + 3Z = 5 \\
-X + 3Y - 2Z = -3 \\
2X - 4Z = 6
\end{cases}
\]

Augmented matrix

\[
\begin{bmatrix}
1 & -4 & 3 & 5 \\
-1 & 3 & -1 & -3 \\
2 & 0 & -4 & 6
\end{bmatrix}
\]

Coefficient matrix

\[
\begin{bmatrix}
1 & -4 & 3 \\
-1 & 3 & -1 \\
2 & 0 & -4
\end{bmatrix}
\]
2. Elementary row operations.

In the previous section, we studied three operations that produce equivalent systems of linear equations:

- Interchange two equations
- Multiply an equation by a nonzero constant
- Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to elementary row operations. An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are row-equivalent when one can be obtained from the other by a finite sequence of elementary row operations.

Elementary row operations:

1. Interchange two rows
2. Multiply a row by a nonzero constant
3. Add a multiple of a row to another row.

Although elementary row operations are relatively simple to perform, they can involve a lot of arithmetic.
Example 2. Elementary row operations.

a. Interchange row 1 and row 2

Original matrix
\[
\begin{bmatrix}
0 & 1 & 3 & 4 \\
-1 & 2 & 0 & 3 \\
2 & -3 & 4 & 1
\end{bmatrix}
\]

New row-equivalent matrix
\[
\begin{bmatrix}
-1 & 2 & 0 & 3 \\
0 & 1 & 3 & 4 \\
2 & -3 & 4 & 1
\end{bmatrix}
\]

b. Multiply the first row by \(\frac{1}{2}\) to produce a new row 1.

Original matrix
\[
\begin{bmatrix}
2 & -4 & 6 & -2 \\
1 & 3 & -3 & 0 \\
5 & -2 & 1 & 2
\end{bmatrix}
\]

New row-equivalent matrix
\[
\begin{bmatrix}
1 & -2 & 3 & -1 \\
0 & 1 & 0 & 0 \\
5 & -2 & 1 & 2
\end{bmatrix}
\]

c. Add -2 times row 1 to the third row to produce a new row 3.

Original matrix
\[
\begin{bmatrix}
1 & 2 & -4 & 3 \\
0 & 3 & -2 & 4 \\
2 & 1 & 5 & -2
\end{bmatrix}
\]

New row-equivalent matrix
\[
\begin{bmatrix}
1 & 2 & -4 & 3 \\
0 & 3 & -2 & 4 \\
0 & -3 & 13 & -8
\end{bmatrix}
\]
Example 3. Using elementary row operations

Linear system
\[
\begin{align*}
\begin{cases}
X - 2y + 3z &= 9 \quad \text{Eqn. 1} \\
-x + 3y &= -4 \quad \text{Eqn. 2} \\
2x - 5y + 5z &= 17 \quad \text{Eqn. 3}
\end{cases}
\end{align*}
\]

Associated augmented matrix
\[
\begin{bmatrix}
1 & -2 & 3 & 9 \\
-1 & 3 & 0 & -4 \\
2 & -5 & 5 & 17
\end{bmatrix}
\]

Add Eqn. 1 to Eqn. 2
\[
\begin{align*}
\begin{cases}
X - 2y + 3z &= 9 \\
y + 3z &= 5 \\
2x - 5y + 5z &= 17
\end{cases}
\end{align*}
\]

Add row 1 to row 2 to produce a new row 2
\[
\begin{bmatrix}
1 & -2 & 3 & 9 \\
0 & 1 & 3 & 5 \\
2 & -5 & 5 & 17
\end{bmatrix}
\]

Add -2 times Eqn. 1 to Eqn. 3
\[
\begin{align*}
\begin{cases}
X - 2y + 3z &= 9 \\
y + 3z &= 5 \\
-y - 2z &= -1
\end{cases}
\end{align*}
\]

Add -2 times row 1 to row 3 to produce a new row 3
\[
\begin{bmatrix}
1 & -2 & 3 & 9 \\
0 & 1 & 3 & 5 \\
0 & -1 & -1 & -1
\end{bmatrix}
\]

Add Eqn. 2 to Eqn. 3
\[
\begin{align*}
\begin{cases}
X - 2y + 3z &= 9 \\
y + 3z &= 5 \\
-y - 2z &= -1
\end{cases}
\end{align*}
\]

Add row 2 to row 3 to produce a new row 3
\[
\begin{bmatrix}
1 & -2 & 3 & 9 \\
0 & 1 & 3 & 5 \\
0 & 0 & 2 & 4
\end{bmatrix}
\]
Multiply Eqn. 3 by $\frac{1}{2}$

\[
\begin{align*}
X - 2Y + 3Z &= 9 \\
Y + 3Z &= 5 \\
2 &= 2
\end{align*}
\]

Multiply row 3 by $\frac{1}{2}$ to produce a new row 3

\[
\begin{bmatrix}
1 & -2 & 3 & 9 \\
0 & 1 & 3 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

$\frac{1}{2}R_3 \rightarrow R_3$

Using back-substitution, we find the solution \((x, y, z) = (1, -2, 3)\) which is unique. \(\square\)

**Remark.**

The last matrix in the above example is in row-echelon form.

A matrix in row-echelon form has the properties below:

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first non-zero entry is 1 (called a leading 1).
3. For two successive (non-zero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.
A matrix in row-echelon form is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

**Gaussian elimination with back-substitution.**

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

**Example 4. Gaussian elimination with back-substitution**

Solve the system

\[
\begin{align*}
    x_2 + x_3 - 2x_4 &= -3 \\
    x_1 + 2x_2 - x_3 &= 2 \\
    2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\
    x_1 - 4x_2 - 7x_3 - x_4 &= -19.
\end{align*}
\]

**Solution.**

The augmented matrix is given by
Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

\[
\begin{bmatrix}
0 & 1 & 1 & -2 & -3 \\
1 & 2 & -1 & 0 & 2 \\
2 & 4 & 1 & -3 & -2 \\
1 & -4 & -7 & -1 & -19 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
2 & 4 & 1 & -3 & -2 \\
1 & -4 & -7 & -1 & -19 \\
\end{bmatrix}
\overset{R_1 \leftrightarrow R_2}{\rightarrow}
\]

\[
\begin{bmatrix}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
0 & 0 & 3 & -3 & -6 \\
1 & -4 & -7 & -1 & -19 \\
\end{bmatrix}
\overset{R_3 + (-2)R_1}{\rightarrow} R_3
\]

\[
\begin{bmatrix}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
0 & 0 & 3 & -3 & -6 \\
0 & -6 & -6 & -1 & -21 \\
\end{bmatrix}
\overset{R_4 + (-1)R_1}{\rightarrow} R_4
\]

\[
\begin{bmatrix}
0 & 0 & 3 & -3 & -6 \\
0 & -6 & -6 & -1 & -21 \\
0 & -6 & -6 & -1 & -21 \\
-20 & -20 & -20 & -20 & -20 \\
\end{bmatrix}
\]
Now that the first column is in the desired form, we can change the second column as shown below

\[
\begin{bmatrix}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
0 & 0 & 3 & -3 & -6 \\
0 & 0 & 0 & -13 & -39
\end{bmatrix} \quad \overset{R_4 + 6R_2 \rightarrow R_4}{\longrightarrow}
\]

To write the third and fourth columns in proper form, multiply the third row by \(\frac{1}{3}\) and the fourth row by \(-\frac{1}{13}\).

\[
\begin{bmatrix}
1 & 2 & -1 & 0 & 2 \\
0 & 1 & 1 & -2 & -3 \\
0 & 0 & 1 & -1 & -2 \\
0 & 0 & 0 & 1 & 3
\end{bmatrix} \quad \overset{\frac{1}{3}R_3 \rightarrow R_3}{\longrightarrow} \quad \overset{-\frac{1}{13}R_4 \rightarrow R_4}{\longrightarrow}
\]

The matrix is now in row-echelon form, and the corresponding system is shown below

\[
\begin{cases}
x_1 + 2x_2 - x_3 = 2 \\
x_2 + x_3 - 2x_4 = -3 \\
x_3 - x_4 = -2 \\
-21 - x_4 = 3
\end{cases}
\]
Use back-substitution to find the solution which is unique: \((x_1, x_2, x_3, x_4) = (-1, 2, 1, 3)\).

Example 5. A system with no solution

Solve the system:

\[
\begin{align*}
    x_1 - x_2 + 2x_3 &= 4 \\
    x_1 + x_3 &= 6 \\
    2x_1 - 3x_2 + 5x_3 &= 4 \\
    3x_1 + 2x_2 - x_3 &= 1
\end{align*}
\]

Solution.
The augmented matrix for this system is

\[
\begin{bmatrix}
    1 & -1 & 2 & 4 \\
    1 & 0 & 1 & 6 \\
    2 & -3 & 5 & 4 \\
    3 & 2 & -1 & 1
\end{bmatrix}
\]

Now, let us apply Gaussian elimination to the augmented matrix:

\[
\begin{bmatrix}
    1 & -1 & 2 & 4 \\
    0 & 1 & -1 & 2 \\
    2 & -3 & 5 & 4 \\
    3 & 2 & -1 & 1
\end{bmatrix}
\begin{align*}
    R_2 & \leftrightarrow R_1 \\
    R_2 & \leftrightarrow R_1
\end{align*}
\]
Note that row 3 of this matrix consists entirely of zeros except for the last entry. This means that the original matrix system of linear equations is inconsistent. To see why this is true, convert back to a system of linear equations.

\[
\begin{align*}
x_1 - x_2 + 2x_3 &= 4 \\
x_2 - x_3 &= 2 \\
0 &= -2 \\
5x_2 - 7x_3 &= -14 \\
-23 &= -23
\end{align*}
\]
The third equation is not possible, so the system has no solution. □

(3) Gauss-Jordan elimination.

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called Gauss-Jordan elimination after Carl Friedrich Gauss and Wilhelm Jordan, continues the reduction process until a reduced row-echelon form is obtained.

Example 6. Gauss-Jordan elimination

Use Gauss-Jordan elimination to solve the system

\[
\begin{align*}
   x - 2y + 3z &= 9 \\
   -x + 3y &= -4 \\
   2x - 5y + 5z &= 17 \\
\end{align*}
\]

Solution.

In Example 3, we used Gaussian elimination to obtain the row-echelon form
\[
\begin{bmatrix}
1 & -2 & 3 & 9 \\
0 & 1 & 3 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

Now, apply elementary row operations until we obtain zeros above each of the leading 1's, as shown below:

\[
\begin{bmatrix}
1 & 0 & 9 & 19 \\
0 & 1 & 3 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix} \rightarrow R_1 + 2R_2 \rightarrow R_1
\]

\[
\begin{bmatrix}
1 & 0 & 9 & 19 \\
0 & 1 & 10 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix} \rightarrow R_2 + (-3)R_3 \rightarrow R_2
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 10 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix} \rightarrow R_1 + (-9)R_3 \rightarrow R_1
\]

The matrix is now in reduced row-echelon form.

Converting back to a system of linear equations, you have \((x, y, z) = (1, -1, 2)\).