YMSC COURSE PROPOSAL: CREATIVE PROBLEM SOLVING, QIUZHEN COLLEGE, TSINGHUA UNIVERSITY, FALL 2022

COURSE SYLLABUS

1. General information

Instructors: Yitwah Cheung (Professor YMSC), Cezar Lupu (Postdoc BIMSA)
Teaching Assistant-Homework Grader: Cezar Lupu
Office: Jing Zhai 220 (Yitwah Cheung), room B2 West Road Yanqi Lake (Cezar Lupu)
Email: yitwah@tsinghua.edu.cn (Yitwah Cheung), lupucezar@bimsa.cn (Cezar Lupu)
Lecture Time:

- Thursday (Lecture), 19:20-20:55 PM
- Friday (Recitation), 19:20-20:55 PM

Location: (Teaching Building 6) 6A115
Office hours: Make an appointment with the instructors via email. The timetable meetings are given below.

- Thursday, 18:00-19:00 PM
- Friday, 18:00-19:00 PM

Credits: 4 credit units
Topics covered: Elementary Algebra, Abstract & Linear Algebra, Real Analysis, Combinatorics, Geometry & Trigonometry, Number Theory, Probability, Differential Equations

2. Course description

This course (YMSC-Creative Problem Solving) teaches important skills in problem solving that are not taught in a systematic way in any other course. These skills are extremely valuable in preparing students for jobs and for graduate-level research. The teaching style will be a mixture of a lecture and a problem-solving session. Training will involve the study of problems from previous Putnam competitions, for which this course can be regarded as a useful preparation. An attempt will be made to look for unifying mathematical ideas. General strategies for solving problems will also be discussed.

By the end of this course, students should develop fundamental problem solving skills, and become accustomed to concentrating on a problem for an extended period of time. This course will be focused on the raw creative problem-solving skills which can serve as an essential ingredient in almost every field of activity.

Last but not least, after students take the Putnam competition, they will have the chance to experience some research oriented talks suitable for undergraduates from various faculty at Tsinghua.

2.1. **The "Shadow" Putnam Competition for students at YMSC, Tsinghua University.** The William Lowell Putnam Mathematical Competition is the premiere competition for undergraduate students in North America. More than 500 universities compete in this contest organized by the Mathematical Association of America (MAA).

The test is supervised by faculty members of each participating university. Every problem is graded on a scale of 0-10. The problems are usually listed in increasing order of difficulty, with A1 and B1 the easiest, and A6 and B6 the hardest. Top 5 scoring students on the Putnam exam are named Putnam Fellows. A student can take this exam maximum four times and the Putnam official team of the university consists of 3 members.

The 83rd Putnam Mathematical Competition, 2022

It will take place on **SUNDAY, DECEMBER 4TH** and it consist of two sessions of three hours each:

Morning Session: 9am-12pm, location: TBD.
Afternoon Session: 2pm-5pm, location: TBD.

William Lowell PUTNAM Mathematical Competition

3. Grading Policy, Grading Scale, Weighted Value of Assignments and Tests

- Homework assignments: **60%** (4 homework assignments in total; you must solve 10-15 problems from each homework to get full credit!)
- Participation in the Putnam competition: **20%** (you must take both sessions of the exam!)
- Performance in the Putnam competition: **20%** (if the participant's score is greater or equal than 30 points which is 3 problems completely solved)

Letter grades will then be assigned in accordance with the following correspondence:

- Letter **grade A** = a percentile grade of 90% of higher
- Letter **grade A-** = a percentile grade of 80% or higher, that is lower than 90%
- Letter **B**= a percentile grade of 70% or higher, that is lower than 80%
- Letter **B-**=a percentile grade of 60% or higher, that is lower than 70%
- Letter **C**=a percentile grade lower than 60%

4. Tentative Schedule of Class Meetings and Topics

Week 1: Elementary algebra (September 15, 16)

This will cover problems on topics such as algebraic identities and inequalities as well as complex numbers, mathematical induction, functional equations and polynomials (integer polynomials, roots of polynomials).

Week 2: Geometry and trigonometry (September 22, 23)

This will cover problems on topics such as vectors, conics, quadratics, and other curves in the plane as well as trigonometric formulas.

Week 3: Combinatorics (September 29, 30)

This will cover problems on topics combinatorial geometry, pigeonhole principle, generating functions graph theory, binomial identities and counting strategies.

Week 4: Probability (October 6, 7)

This will cover problems on continuous random variables. Jointly continuous random variables, independence, conditioning, functions of one or more random variables, change of variables. Examples including some with later applications in statistics.

Week 5: Number Theory I (October 13, 14)

This will cover problems on topics such as integer-valued sequences and functions, congruences, divisibility and arithmetic functions.

Week 6: Number theory II (October 20, 21)

This will cover problems on topics such as quadratic residues, diophantine equations, and algebraic & analytic methods in number theory.

Week 7: Abstract algebra (October 27, 28)

This will cover problems on topics such as groups, rings, and finite fields.

Week 8: Linear algebra I (November 3, 4)

This will cover topics on 2×2 and 3×3 matrices and determinants. Also, will cover problems on topics such as vectors spaces, linear transformations, characteristic and minimal polynomials, eigenvalues and eigenvectors.

Week 9: Linear algebra II (November 10, 11)

This will cover some special topics in linear algebra such as special classes of diagonalizable matrices, Jordan canonical form, Schur triangularization form, spectral theorem for normal matrices.

Week 10: Real analysis I (November 17, 18)

This will cover problems on topics such as sequences and series of real numbers. Moreover, we will also cover problems on topics such as limits of functions and continuity.

Week 11: Real analysis II (November 24, 25)

This will cover problems on topics such as differentiability (mean value theorems, Taylor series, etc) and integrability (Riemann integrals, continuity of integrals, integral inequalities). Also, we plan to cover sequences and series of functions.

Week 12: Real analysis III (December 1, 2)

This will cover problems on multivariable differential and integral calculus. This will include extrema of functions, Lagrange multipliers, Fubini's theorem, change of variables formulas and divergence theorem.

Week 13: The 2022 Putnam exam discussion (December 8, 9)

This will cover the discussion over the problems given at the Putnam exam.

Week 14: Invited lectures I & II

- Lecture I: The anatomy of a Putnam problem
  Speaker: Cezar Lupu
  Date: December 15 (Thursday)

  Abstract: In this talk, I shall bring into perspective an analysis problem given at the Putnam Mathematical Competition in 2007. The question B2 on the exam reads as the following:

  Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left|\int_0^\alpha f(x)\,dx\right| \leq \frac{1}{8}\sup_{0\leq x\leq 1}|f'(x)|.$$

Following Polya's approach on how to solve a problem, we "dissect" the above question by giving several solutions and by relating it to other similar problems or problems with similar ideas. In fact, we go back to one of Polya's original problems and show how ideas can be applied in our situation. Last but not least, we also discuss some variations of our main question.

- Lecture II: An elementary problem equivalent to the Riemann hypothesis
  Speaker: Cezar Lupu
  Date: December 16 (Friday)

Abstract: In this lecture, I will speak about my absolute favorite paper from the American Mathematical Monthly which is about an elementary problem equivalent to the infamous Riemann hypothesis which concerns the complex (nontrivial) zeros of the Riemann zeta function. This elementary problem of Lagarias concerns the sum of divisors function and the harmonic number. If time allows, I will talk about Robin's inequality and superabundant numbers.

Week 15: Invited lectures I & II

- Lecture I: Continued fraction of a real number
  Speaker: Yitwah Cheung
  Date: December 22 (Thursday)

Abstract: In this lecture, I will give a brief introduction to the theory of continued fractions, starting with the Gauss map and explaining two theorems of Lagrange, the first identifying quadratic irrationals as those real numbers that have eventually periodic continued fraction and the second characterizing the convergents of the continued fraction as best rational approximations of the second kind.

- Lecture II: Geometry of continued fractions
  Speaker: Yitwah Cheung
  Date: December 23 (Friday)

Abstract: In this lecture, I will explain concretely how the Galois embedding of a totally real number field maps the ring of integers to a lattice inside some euclidean space. We will prove a result of Galois showing that periodic orbits of the Gauss map are parametrized by reduced quadratic irrationals. We shall interpret the group of units as the stabilizer of a Lie group acting on a “moduli" space of lattices.

Week 16: Invited lectures III & IV

- Lecture III: Dynamics of continued fractions
  Speaker: Yitwah Cheung
  Date: December 29 (Thursday)

  Abstract: In this lecture, we discuss the concepts of first return, geodesic flow and modular surface, and use them to show that the Gauss map is essentially the first return to a certain transversal to the geodesic flow of the modular surface. We give a brief introduction to the idea of Dani correspondence.

- Lecture IV: Distribution of Farey sequences
  Speaker: Yitwah Cheung
  Date: December 30 (Friday)

  Abstract: In the previous lectures, we saw how the continued fraction of a real number essentially encodes the evolution of a unimodular lattice under the action of a diagonal flow. In this lecture, we explain how replacing the dynamics on the moduli space with a unipotent flow, one recovers the so-called BCZ map, which encodes the distribution of Farey sequences. Using a classical result of Franel and Landau, we give a characterization of the Riemann Hypothesis in terms of the BCZ map.

References

[1] R. Gelca, T. Andreescu, *Putnam and Beyond*, Springer Verlag, 2007.
[2] K. Kedlaya, B. Poonen, R. Vakil, *The William Lowell Putnam Mathematical Competition 1985-2000: Problems, Solutions and Commentary*, The Mathematical Association of America, Washington, D.C., 2002.
[3] K. Kedlaya, The Putnam Archive (1985-2021).
[4] L. Larson, Problem-Solving Through Problems, Springer Verlag, 1983.